44 QUANTUM FIELD THEORY AND CRITICAL PHENOMENA: EXERCISES

Most of these exercises were included in the third edition and have been removed from the fourth edition for lack of space. The references to equations, chapters and sections correspond to the fourth edition.

44.1 Algebraic Preliminaries : Chapter 1

Exercise 1.1

Prove, using the method of fermion integration

$$\det\left[a_{ij}+s_i\delta_{ij}\right]=\det a_{ij}\left[\sum_n \frac{1}{n!}\sum_{i_1,i_2,\ldots,i_n} s_{i_1}s_{i_2}\ldots s_{i_n}\det a^{-1}_{i_l i_k}\right]. \tag{44.1}$$

Solution. One considers the fermion integral

$$\int\left(\prod_i \mathrm{d}\theta_i \mathrm{d}\bar\theta_i\right)\exp\left(\sum_{i,j=1}^{n} a_{ij}\bar\theta_i\theta_j+\sum_i s_i\bar\theta_i\theta_i\right).$$

Integration directly yields the l.h.s. of equation (44.1). Expanding in perturbation theory the term proportional to the s_i's and using fermion's Wick theorem (1.77) yields the r.h.s..

Exercise 1.2

Random hermitian matrices in the limit of large size provide an interesting model for complex hamiltonians (see Appendix A22). A simple example is provided by matrices with a gaussian probability distribution (the Gaussian Unitary Ensemble or GUE).

Of special interest is the spectrum of such matrices. One intends to calculate, here the expectation value with gaussian weight of the characteristic polynomial of a hermitian matrix .

One thus considers random hermitian $N\times N$ matrices with the distribution $\mathrm{d}\rho(M)$:

$$\mathrm{d}\rho(M)=\mathcal{N}^{-1}\mathrm{d}^{N^2}M\ \mathrm{e}^{-N\,\mathrm{tr}\,M^2/2},\quad \mathrm{d}^{N^2}M\equiv\prod_i \mathrm{d}M_{ii}\prod_{i<j}\mathrm{d}\,\mathrm{Re}\,M_{ij}\,\mathrm{d}\,\mathrm{Im}\,M_{ij}, \tag{44.2}$$

the constant $\mathcal{N}$ being determined by the condition that the expectation value of 1 is 1

$$\mathcal{N}=\int \mathrm{d}^{N^2}M\ \mathrm{e}^{-N\,\mathrm{tr}\,M^2/2}.$$

The measure (44.2) is invariant under unitary transformations, $M\mapsto U^\dagger MU$, and thus gives the same weight to all matrices that have the same spectrum. We denote the expectation value of a function $F(M)$ with measure (44.2)

$$\langle F(M)\rangle=\mathcal{N}^{-1}\int \mathrm{d}^{N^2}M\ \mathrm{e}^{-N\,\mathrm{tr}\,M^2/2}F(M).$$

(i) Represent the characteristic polynomial of a hermitian matrix M $(\det(M-z))$ as a Grassmann integral.

(ii) Calculate then the expectation value $\mathcal{H}_N(z) = \langle \det(M-z) \rangle$.

(iii) The integrand takes the form of the exponential of a quartic polynomial in Grassmann variables. Note that it can be rewritten as a simple gaussian integral (one variable) of the exponential of a quadratic form in the Grassmann variables.

(iv) Integrate over the Grassmann variables: the final result takes the form of a simple integral. Since one shows quite generally that the expectation value of $\det(M-z)$ with a measure $\mathrm{e}^{-\operatorname{tr} V(M)} \mathrm{d}M$ is the orthogonal polynomial of degree N with measure $\mathrm{e}^{-V(z)} \mathrm{d}z$, one has obtained an integral representation of Hermite polynomials.

(vi) Generalize the method to calculate the expectation value of $[\det(M-z)]^n$.

Solution.

$$\det(M-z) = \int \prod_i \mathrm{d}\bar{\theta}_i \mathrm{d}\theta_i \, \mathrm{e}^{\operatorname{tr} XM - z \operatorname{tr} X},$$

with

$$X_{ij} = \bar{\theta}_i \theta_j .$$

Therefore

$$\operatorname{tr} X^2 = \bar{\theta}_i \theta_j \bar{\theta}_j \theta_i = -\left(\sum_i \bar{\theta}_i \theta_i\right)^2 = -(\operatorname{tr} X)^2.$$

One then notices

$$\mathrm{e}^{\operatorname{tr} X^2/2N} = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}s \ \mathrm{e}^{-s^2/2 + is \sum_i \bar{\theta}_i \theta_i / \sqrt{N}} .$$

It is convenient to change $s \mapsto s\sqrt{N}$. The integral over Grassmann variables is immediate

$$\mathcal{H}_N(z) = \sqrt{N/2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s \ \mathrm{e}^{-Ns^2/2} (is - z)^N . \tag{1}$$

A similar method allows to calculate the expectation value of $[\det(M-z)]^n$. One has to introduce a hermitian matrix S $n \times n$ and the expression is replaced by

$$\langle [\det(M-z)]^n \rangle = (N/2\pi)^n/2 \int \mathrm{d}S \ \mathrm{e}^{-N \operatorname{tr} S^2/2} \left[\det(iS - z)\right]^N .$$

Exercise 1.3

In associative algebras one can define a differentiation operation by replacing Leibnitz's rule by:

$$\mathrm{D}(AB) = \mathrm{D}A\, B + H(A) \mathrm{D}B . \tag{44.3}$$

Show that the condition that H is a homomorphism of algebra is a necessary condition for the consistency of the definition by calculating $\mathrm{D}(ABC)$ in two different ways.

Solution. First

$$\begin{aligned} \mathrm{D}(ABC) &= \mathrm{D}A\, BC + \mathrm{H}(A) \mathrm{D}(BC) \\ &= \mathrm{D}A\, BC + \mathrm{H}(A)\mathrm{D}B\, C + \mathrm{H}(A)\mathrm{H}(B)\mathrm{D}C . \end{aligned}$$

Also

$$\begin{aligned} \mathrm{D}(ABC) &= \mathrm{D}(AB)C + \mathrm{H}(AB)\mathrm{D}C \\ &= \mathrm{D}A\, BC + \mathrm{H}(A)\mathrm{D}B\, C + \mathrm{H}(AB)\mathrm{D}C . \end{aligned}$$

Comparing both results one finds

$$\mathrm{H}(A)\mathrm{H}(B) = \mathrm{H}(AB).$$

Exercise 1.4

Consider a graded algebra $\mathfrak{A}$ on $\mathbb{C}$ with one generator x (and thus commutative) satisfying the nilpotency condition $x^n = 0$. Construct a differential operator D linear, decreasing the degree by one unit, and consistent with this condition.

Solution. It is of course sufficient to characterize the action of D on x^p. We can set $\mathrm{D}x = 1$. We have to use the basic relation (44.3)

$$\mathrm{D}(AB) = \mathrm{D}A\,B + H(A)\mathrm{D}B\,,$$

where H being a homomorphism of algebra, consistent with the grading is characterized by a complex constant h

$$H(x) = hx\,,$$

while

$$\mathrm{D}(x^p) = c_p x^{p-1}.$$

It is now easy to verify that the solution is

$$\mathrm{D}(x^p) = \frac{1-h^p}{1-h} x^{p-1}.$$

Then the consistency condition $\mathrm{D}(x^n) = 0$ implies $h^n = 1$ with $h \neq 1$.

Note that $\mathrm{D}^n = 0$ and also in the sense of operators acting on $\mathfrak{A}$

$$\mathrm{D}x - hx\mathrm{D} = 1\,.$$

Finally a linear integration operator on $\mathfrak{A}$ satisfying

$$\mathrm{DI} = \mathrm{ID} = 0\,,$$

is

$$\mathrm{I} = \mathrm{D}^{n-1}.$$

Exercise 1.5

Consider an associative graded algebra on $\mathbb{C}$ with generators x_i such that

$$x_i^n = 0\,, \tag{44.4}$$

and satisfying the commutation relation

$$x_j x_i = c x_i x_j \qquad \text{for } j > i\,,$$

where c is a complex number.

Show that a necessary and sufficient condition for the property (44.4) to be independent on the basis in the vector space of generators is $c^n = 1$ with $c \neq 1$. In other words express the condition

$$\left(\sum a_i x_i\right)^n = 0 \qquad \forall a_i \neq \{0, 0, \ldots, 0\}.$$

Solution. We first express the condition for any pair of generators. It is then sufficient to consider the sum of any two of them like x_1 and x_2. The result relies on the inductive proof of the identity:

$$(x_1+x_2)^p=\sum_{m=0}^{p} x_1^m x_2^{p-m}\prod_{l=0}^{m-1}\frac{1-c^{p-l}}{1-c^{l+1}}.$$

Specializing this result for $p=n$ we obtain the necessary condition.

We then note that any linear combination of x_1 and x_2 also satisfies

$$x_3(a_1x_1+a_2x_2)=c(a_1x_1+a_2x_2)x_3\,,\quad \text{and}\quad (a_1x_1+a_2x_2)^n=0\,,$$

which is the condition (44.4). The result thus holds for the pair $\{(a_1x_1+a_2x_2),x_3\}$, and thus any linear combination of three generators.

We can then prove the general result by induction on the number of generators.

44.2 Euclidean Path Integrals in Quantum Mechanics : Chapter 2

Exercise 2.1

2.1.1. Combining equations (2.47,2.45) and (2.62), calculate the energy levels E_n of the hamiltonian H:

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}q^2+\lambda q^4,$$

to first order in λ, from the expression of the partition function $\mathcal{Z}$:

$$\mathcal{Z}=\operatorname{tr}\mathrm{e}^{-\beta H}=\sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_n}\,.$$

Solution. One finds:

$$\mathcal{Z}=\left(2\sinh\left(\beta/2\right)\right)^{-1}\left(1-3\lambda\int_{-\beta/2}^{\beta/2}\mathrm{d}t\,K^2\left(t,t\right)+O\left(\lambda^2\right)\right),$$

in which:

$$K(t,t)=\tfrac{1}{2}\text{cotanh}\,\left(\beta/2\right).$$

Therefore:

$$\mathcal{Z}=\left[2\sinh\left(\beta/2\right)\right]^{-1}\left[1-\frac{3}{4}\lambda\beta\left(\text{cotanh}^2\left(\beta/2\right)\right)+O\left(\lambda^2\right)\right].\tag{44.5}$$

Using two algebraic identities:

$$\sum_{n=0}^{\infty}\exp\left[-\left(n+\tfrac{1}{2}\right)\beta\right]=1/\left[2\sinh\left(\beta/2\right)\right],$$

and, differentiating twice with respect to β,

$$\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)^2\exp\left[-(n+\tfrac{1}{2})\beta\right]=\frac{2\,\text{cotanh}^2(\beta/2)-1}{8\sinh(\beta/2)},$$

one can verify that $\mathcal{Z}$ is the expansion to first order in λ of $\sum \exp\left(-\beta E_n\right)$ with:

$$E_n = n + \tfrac{1}{2} + \tfrac{3}{2}\lambda\left(n^2 + n + \tfrac{1}{2}\right) + O\left(\lambda^2\right). \tag{44.6}$$

2.1.2. Evaluate the contribution of order λ^2 to E_n.

Solution. Setting $\nu = n + \frac{1}{2}$, the result can be written

$$E = \nu + \tfrac{3}{2}\left(\nu^2 + \tfrac{1}{4}\right)\lambda - \tfrac{1}{4}\nu\left(17\nu^2 + \tfrac{67}{4}\right)\lambda^2 + O\left(\lambda^3\right).$$

Note that more generally for the potential

$$V(q) = \tfrac{1}{2}q^2 + v_1 q^4 + v_2 q^6 + O\left(q^8\right),$$

one finds

$$n + \tfrac{1}{2} = E - v_1\left(\tfrac{3}{2}E^2 + \tfrac{3}{8}\right) + E^3\left(\tfrac{35}{4}v_1^2 - \tfrac{5}{2}v_2\right) + E\left(\tfrac{85}{16}v_1^2 - \tfrac{25}{8}v_2\right).$$

2.1.3. Calculate to order λ the energy levels for the potentials of the form:

$$V(q) = \tfrac{1}{2}q^2 + \lambda q^N, \qquad N = 6, 8.$$

Solution. For $N = 6$ and $N = 8$ the results are respectively

$$\begin{aligned} E_n &= n + \tfrac{1}{2} + \tfrac{5}{2}\lambda(n + \tfrac{1}{2})\left(n^2 + n + \tfrac{3}{2}\right) \\ E_n &= n + \tfrac{1}{2} + \tfrac{35}{8}\lambda\left(n^4 + 2n^3 + 5n^2 + 4n + \tfrac{3}{2}\right). \end{aligned}$$

Exercise 2.2

Compare the exact expression (2.28) of the statistical operator of the harmonic oscillator (set for convenience $\hbar = m = \omega = 1$) with the approximate expression obtained by using equation (2.15) and calculating explicitly the integrals in (2.16). The analysis is simplest when $V(q)$ in equation (2.15) is replaced by an equivalent symmetric function of q and q', like $V[(q+q')/2]$. This amounts to introduce the kernel $K(q,q')$,

$$K(q,q') = \sqrt{a/\pi}\,\exp\left[-\tfrac{1}{2}a\left(q^2 + q'^2\right) + bqq'\right], \tag{44.7}$$

with

$$a = \frac{1}{\varepsilon} + \lambda\varepsilon\,, \qquad b = \frac{1}{\varepsilon} + \varepsilon(\lambda - \tfrac{1}{2}),$$

where λ is a free parameter and $\varepsilon = \beta/n$.

Different equivalent discretizations give different values for λ. Discuss the precision as a function of λ.

Solution. It is convenient to parametrize the coefficients a, b as:

$$a = \rho\frac{\cosh\theta}{\sinh\theta}\,, \quad b = \frac{\rho}{\sinh\theta} \quad \Rightarrow \rho = \sqrt{a^2 - b^2}\,, \quad \theta = \ln\left(\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}\right).$$

Expanding in ε one finds

$$\begin{aligned} \rho &= 1 + \tfrac{1}{2}(\lambda - \tfrac{1}{4})\varepsilon^2 + O\left(\varepsilon^4\right) \\ \theta &= \varepsilon\left[1 - \tfrac{1}{2}(\lambda - \tfrac{5}{12})\varepsilon^2\right] + O\left(\varepsilon^5\right). \end{aligned}$$

One verifies that $\mathbf{K}(\theta)\mathbf{K}(\theta') = \mathbf{K}(\theta + \theta')$ and therefore $\mathbf{K}^p(\theta) = \mathbf{K}(p\theta)$ where $\mathbf{K}(\theta)$ is the operator with kernel $K(q,q')$ and parameter θ.

Comparing $K^n(q,q')$ with the exact result (2.28) one sees that the corrections come from the deviations of ρ from 1 and $n\theta$ from β, and are of order $1/n^2$. No value of λ cancels both corrections. A good compromise is $\lambda = 1/3$ which minimizes both corrections. We note that this value corresponds to the result (2.14) obtained from the small ε expansion.

44.3 Path Integrals in Quantum Mechanics: Generalization : Chapter 3

Exercise 3.1

Calculate the energy levels of the $O(\nu)$ invariant hamiltonian

$$H = \tfrac{1}{2}\mathbf{p}^2 + \tfrac{1}{2}\mathbf{q}^2 + \lambda|\mathbf{q}|^4,$$

in which $\mathbf{p}$ and $\mathbf{q}$ are vectors in $\mathbb{R}^\nu$, at order λ. Note that since levels are degenerate at leading order, it is not convenient to calculate only $\operatorname{tr} \mathrm{e}^{-\beta H}$ as in the $\nu = 1$ case (result (44.6)).

Solution. For level n and angular momentum l, setting $\mu = l + \nu/2 - 1$, one finds

$$E_{nl} = 2n + 1 + \mu + \tfrac{1}{2}\lambda\left[3(2n+1)^2 + 6\mu(2n+1) + 2\mu^2 + 1\right] + O\left(\lambda^2\right).$$

Exercise 3.2

3.2.1. Using expression (3.18), calculate the matrix elements of the statistical operator $U(\beta) = \mathrm{e}^{-\beta H}$ for $\mathbf{A}(\mathbf{q}) = \frac{1}{2}\mathbf{B} \times \mathbf{q}$, $\mathbf{B}$ constant, i.e. for a constant magnetic field, and $V(\mathbf{q}) = 0$.

Solution. We first consider the effect of a translation on the evolution operator. In the path integral we change variables, setting

$$\mathbf{q}(t) \mapsto \mathbf{q}(t) + \mathbf{a},$$

where $\mathbf{a}$ is a constant vector. The boundary conditions then become

$$\mathbf{q}(-\beta/2) = \mathbf{q}' - \mathbf{a}\,, \quad \mathbf{q}(\beta/2) = \mathbf{q}'' - \mathbf{a}\,, \qquad \beta = t'' - t'\,.$$

The action receives an additive contribution $\frac{1}{2}ie\mathbf{B}\times\mathbf{a}(\mathbf{q}''-\mathbf{q}')$ due to the magnetic field, which leads to the relation

$$\langle \mathbf{q}'' | U(\beta) | \mathbf{q}' \rangle = \exp[\tfrac{1}{2}ie(\mathbf{B}\times\mathbf{a})(\mathbf{q}''-\mathbf{q}')]\,\langle \mathbf{q}''-\mathbf{a} | U(\beta) | \mathbf{q}'-\mathbf{a} \rangle\,.$$

We can use this relation for $\mathbf{a} = \mathbf{q}'$ to simplify calculations. The additional phase then becomes $\frac{1}{2}ie\mathbf{B}\cdot(\mathbf{q}'\times\mathbf{q}'')$.

It remains to calculate $\langle \mathbf{q}''-\mathbf{q}' | U(\beta) | \mathbf{0} \rangle$.

We thus solve the classical equation of motion. The motion along $\mathbf{B}$ being free, we restrict ourselves to the plane perpendicular to $\mathbf{B}$. We find

$$\mathcal{S}_\mathrm{c} = \frac{eB}{4\tanh(eB\beta/2m)}\left(\mathbf{q}''-\mathbf{q}'\right)^2.$$

We then evaluate the determinant resulting from the gaussian integration. We have to calculate the eigenvalues of the differential operator

$$-m\frac{\mathrm{d}^2}{(\mathrm{d}t)^2}\delta_{ij} + ieB_k\epsilon_{kij}\frac{\mathrm{d}}{\mathrm{d}t},$$

with the boundary conditions that the eigenfunctions vanish at $\pm\beta/2$. After Fourier transformation this operator takes the form

$$\begin{pmatrix} m\omega^2 & -\omega eB \\ \omega eB & m\omega^2 \end{pmatrix},$$

where ω is the corresponding frequency. The eigenvalues λ_n then follow:

$$\lambda_n = n^2\pi^2/\beta^2 + e^2B^2/4m^2\,,$$

each eigenvalue being twice degenerate. The product of eigenvalues divided by the value in zero field then follows from the identity

$$\det\left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega^2\right)\Big/\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\right] = \prod_{n>0}\left(1+\frac{\omega^2\beta^2}{n^2\pi^2}\right) = \frac{\sinh\omega\beta}{\omega\beta}. \tag{44.8}$$

Normalizing with respect to the free hamiltonian one finally obtains:

$$\langle \mathbf{q}''\,|U(\beta)|\,\mathbf{q}'\rangle = \frac{eB}{4\pi\sinh(\beta eB/2m)}\exp\left[\tfrac{1}{2}ie\mathbf{B}\cdot(\mathbf{q}'\times\mathbf{q}'')\right]\exp[-\mathcal{S}_\mathrm{c}].$$

3.2.2. Add to the previous action an harmonic potential term $V(\mathbf{q}) = \frac{1}{2}m\omega^2 q^2$ and calculate again the evolution operator.

Solution. The classical action then becomes

$$\begin{aligned}\mathcal{S}_\mathrm{c} = \frac{m\omega'}{2\sinh\beta\omega'}&\left[(\mathbf{q}'^2+\mathbf{q}''^2)\cosh\beta\omega' - 2\mathbf{q}'\mathbf{q}''\cosh\left(\frac{eB\beta}{2m}\right)\right.\\ &\left.-2i\sinh\left(\frac{eB\beta}{2m}\right)\hat{\mathbf{B}}\cdot(\mathbf{q}'\times\mathbf{q}'')\right],\end{aligned}$$

where

$$\hat{\mathbf{B}} = \mathbf{B}/B\,,\qquad \omega' = \sqrt{\omega^2+e^2B^2/4m^2}\,.$$

The gaussian integration then yields the factor $\mathcal{N}$

$$\mathcal{N} = \frac{eB}{4\pi\sinh(\beta\omega')}.$$

Exercise 3.3

Discuss, in the free case, the change of variables $q = f(q')$ in the path integral (2.19). It is suggested to return to discrete forms (2.16,2.17) and then to compare with the expressions (3.21,3.22).

Solution. If we start from the discretized free evolution operator we obtain of course the action

$$\mathcal{S}_1 = \sum_k \frac{1}{2\varepsilon}\left[f(q_{k+1})-f(q_k)\right]^2,$$

with a measure $\prod_k f'(q_k)dq_k$.

If we change variables in the continuum path integral we find the action

$$\mathcal{S} = \tfrac{1}{2}\int \mathrm{d}t\, \dot{q}^2(t) f'^2\bigl(q(t)\bigr),$$

which we could imagine discretizing in a hermitian way as

$$\mathcal{S}_2 = \sum_k \frac{1}{2\varepsilon}(q_{k+1}-q_k)^2 f'^2\bigl((q_{k+1}+q_k)/2\bigr),$$

with the same measure as previously.

Let us thus calculate the difference between the terms in $\mathcal{S}_2$ and $\mathcal{S}_1$. We find

$$\mathcal{S}_1 - \mathcal{S}_2 = \sum_k \tfrac{1}{12\varepsilon}(q_{k+1}-q_k)^4 f'(q_k) f'''(q_k).$$

We that since $q_{k+1}-q_k = O(\varepsilon^{1/2})$ this term is of order ε, i.e. comparable to a potential term. Returning to the initial variables and treating this difference at first order in perturbation we find that the two actions differ by the potential

$$\mathcal{S}_2 - \int \mathrm{d}t\, \dot{q}^2(t) \sim -\int \mathrm{d}t\, f'''(q(t))[f'(q(t))]^{-3},$$

which exhibits the difference in the order of operators in the two quantization methods.

44.4 Stochastic Differential Equations : Chapter 4

Exercise 4.1

Study of the Langevin equation:

$$\dot{x} = -\tfrac{1}{2}ax^n(t) + \nu(t) \tag{44.9}$$

in which $\nu(t)$ is the gaussian white noise considered in equation (4.3) and a is a parameter.

4.1.1. Establish coupled differential equations for the moments $\langle x^k(t)\rangle$ using the method which leads to the Fokker–Planck equation (Section 4.3). Deduce the possible form of the moments at equilibrium.

Solution. To derive coupled differential equations for moments we differentiate the moments with respect to time and use the equation.

$$\dot{x} = -\tfrac{1}{2}ax^n(t) + \nu(t). \tag{44.10}$$

Setting $\left\langle x^k\right\rangle = m_k$ we get

$$\dot{m}_k = -\tfrac{1}{2}akm_{k+n-1} + k\left\langle x^{k-1}(t)\nu(t)\right\rangle.$$

To evaluate the last term we use the same strategy as in the derivation of the Fokker–Planck equation in Section 4.3 based on time discretization, and find

$$\left\langle x^{k-1}(t)\nu(t)\right\rangle = \tfrac{1}{2}\Omega(k-1)m_{k-2}.$$

Therefore

$$\dot{m}_k = -\tfrac{1}{2}akm_{k+n-1} + \tfrac{1}{2}k(k-1)\Omega m_{k-2}.$$

In particular if we assume the existence of an equilibrium we find the relations

$$m_{k+n-1} = (k-1)\Omega m_{k-2}/a\,,$$

which determines all moments once m_k, $k=1,...,n$ are known.

4.1.2. Write the Fokker–Planck equation and study the existence of the equilibrium distribution.

Solution. The Fokker–Planck reads:

$$\dot{P}(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\left[\Omega\frac{\partial P}{\partial x} + ax^n P\right]. \tag{44.11}$$

The possible equilibrium distribution is the static solution

$$P(x) \propto \exp\left(-\frac{a}{\Omega(n+1)}x^{n+1}\right).$$

For $P(x)$ to be normalizable n must be odd and a positive. Transforming the FP hamiltonian into a hermitian hamiltonian one verifies it has a complete discrete spectrum and the convergence towards equilibrium is exponential.

4.1.3. What can be said if the parameter a and $x(t)$ are complex and $\nu(t)$ is still the real noise (4.2). Begin with $n=1$; for $n>2$ this is a research problem.

For $n=1$ we can solve the Langevin equation

$$x(t) = x(0)\,\mathrm{e}^{-at/2} + \int_0^t \mathrm{d}t'\,\nu(t')\,\mathrm{e}^{a(t'-t)/2}\,.$$

It is clear that the condition for the existence of an equilibrium distribution is $\operatorname{Re} a > 0$. Then all averages are determined by the two-point function.

Exercise 4.2

The goal of this exercise is to compare the properties of the ordinary Langevin equation with a second order stochastic differential equation. Consider thus the dynamical equation

$$a\ddot{x} + b\dot{x} + cx = \nu(t),$$

in which a, b and c are constants, and $\nu(t)$ still a gaussian white noise. Determine the equilibrium distribution, the relaxation time....

Solution. The solution of the stochastic differential equation can be written

$$x(t) = A_0\,\mathrm{e}^{-r_1 t} + B_0\,\mathrm{e}^{-r_2 t} + \frac{1}{a\,(r_1 - r_2)}\int_0^t \mathrm{d}\tau\left(\mathrm{e}^{-r_1(\tau-t)} - \mathrm{e}^{-r_2(\tau-t)}\right)\nu(\tau),$$

where r_1, r_2 are the solutions of the equation

$$ar^2 - br + c = 0\,.$$

First it is clear that an equilibrium can be reached only if both r_1, r_2 have a positive real part, which implies $b/a > 0$, $c/a > 0$. Then at large times we can forget about the boundary conditions at $t = 0$, or equivalently we can integrate the equation from $t = -\infty$.

1. The two-point function. Assuming that the gaussian noise distribution is normalized as

$$\langle \nu(t)\nu(t')\rangle = 2\delta(t-t'),$$

we can calculate the two-point correlation function:

$$\langle x(t)x(t')\rangle = \frac{1}{a^2\left(r_2^2 - r_1^2\right)}\left(\frac{1}{r_1}\,\mathrm{e}^{-r_1|t-t'|} - \frac{1}{r_2}\,\mathrm{e}^{-r_2|t-t'|}\right). \tag{44.12}$$

Note that here, in contrast with the ordinary Langevin equation, also the $\langle \dot{x}(t)\dot{x}(t')\rangle$ correlation function is continuous at $t = t'$ and thus the typical paths are smoother.

2. The equilibrium distribution. The equilibrium distribution is gaussian in the two variables $x(t)$ and $\dot{x}(t)$. It is thus characterized by

$$\begin{aligned}\left\langle x^2(t)\right\rangle &= 1/\left[a^2 r_1 r_2\left(r_1+r_2\right)\right] = 1/(bc),\\ \left\langle \dot{x}^2(t)\right\rangle &= 1/\left[a^2\left(r_1+r_2\right)\right] = 1/(ab).\\ \langle \dot{x}(t)x(t)\rangle &= 0\,.\end{aligned}$$

3. The auto-correlation time. We observe that there are two possible situations:

(i) Both r_1 and r_2 are real and then at large times the largest value gives negligible contributions. The second order differential equation can be well approximated by an ordinary Langevin equation. The auto-correlation time τ is given by

$$\tau = \frac{b+\sqrt{b^2-4ac}}{2c}.$$

(ii) r_1 and r_2 are complex conjugated. Both terms then contribute and the behaviour differs from the ordinary Langevin equation. The auto-correlation time τ is then

$$\tau = 2a/b\,.$$

(iii) The border case $r_1 = r_2$ is specially interesting and corresponds to *critical damping*. At b, c fixed (and thus also $\left\langle x^2(t)\right\rangle$) it leads to the fastest decay of correlation or relaxation towards equilibrium:

$$\tau_{\min} = \tfrac{1}{2}(b/c) = \tfrac{1}{2}\tau(a=0).$$

44.5 Path and Functional Integrals in Quantum Statistical Physics : Chapter 5

Exercise 5.1

As suggested in Section 5.2 evaluate the path integral corresponding to the quadratic action (5.27) with open boundary conditions $z(t'') = z$", $\bar{z}(t') = \bar{z}'$ and infer the result (5.29).

Solution. We first write the equation of motion

$$\begin{aligned} -\dot{z}(t) + \omega z(t) &= b(t), \\ \dot{\bar{z}}(t) + \omega \bar{z}(t) &= \bar{b}(t). \end{aligned}$$

The solutions are

$$\begin{aligned} z(t) &= z'' \, \mathrm{e}^{\omega(t-t'')} + \int_t^{t"} \mathrm{d}\tau \; \mathrm{e}^{\omega(t-\tau)} \, b(\tau), \\ \bar{z}(t) &= \bar{z}' \, \mathrm{e}^{\omega(t'-t)} + \int_{t'}^{t} \mathrm{d}\tau \; \mathrm{e}^{\omega(\tau-t)} \, \bar{b}(\tau). \end{aligned}$$

Taking into account the equation of motion the classical action simplifies

$$\begin{aligned} \mathcal{S}_c &= -\bar{z}(t')z(t') - \int_{t'}^{t''} \bar{b}(t)z(t)\mathrm{d}t \\ &= -\bar{z}' z'' \, \mathrm{e}^{\omega(t'-t'')} - \bar{z}' \int_{t'}^{t"} \mathrm{d}\tau \; \mathrm{e}^{\omega(t'-t)} \, \bar{b}(t) - z'' \int_{t'}^{t''} \mathrm{d}t \, \bar{b}(t) \, \mathrm{e}^{\omega(t-t'')} \\ &\quad - \int_{t'}^{t''} \mathrm{d}t \, \bar{b}(t) \int_t^{t"} \mathrm{d}\tau \; \mathrm{e}^{\omega(t-\tau)} \, b(\tau). \end{aligned}$$

After shifting $\bar{z}(t), z(t)$ by the classical solution we have to perform a gaussian integration which depends only on t', t''. It yields a normalization which can be directly inferred from the ground state energy. Taking the trace (expression (5.13)) one recovers the expression (5.29).

Exercise 5.2

Use the holomorphic representation to calculate the order λ correction to the energy levels of the hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \lambda q^4 .$$

Perform the calculation for the two different path integral representations, obtained either by directly replacing the classical phase space variables p, q by the complex variables $z, \bar{z}$, or by writing the hamiltonian in terms of creation and annihilation operators, commuting all creation operators to the left, and then only replacing quantum by classical variables.

Solution. In the first case the interaction term yields a contribution to the action

$$\tfrac{1}{4}\lambda \int \mathrm{d}t \, \left(\bar{z}(t) + z(t)\right)^4 .$$

At this order only the term proportional to $(\bar{z}z)^2$ contributes. Therefore, using Wick's theorem, we find

$$\operatorname{tr} \mathrm{e}^{-\beta H} = \frac{1}{2\sinh(\omega\beta/2)} \left[1 - 3\lambda \int \mathrm{d}t \, \langle \bar{z}(t) z(t) \rangle^2 + O\left(\lambda^2\right)\right],$$

where $\langle \bar{z}(t) z(t) \rangle$ is given by the propagator (5.30) at time 0,

$$\langle \bar{z}(t) z(t) \rangle \equiv \Delta(0) = \frac{1+\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}} \theta(0).$$

With the convention $\theta(0) = \frac{1}{2}$ we find

$$\operatorname{tr} \mathrm{e}^{-\beta H} = \frac{1}{2\sinh(\omega\beta/2)} \left[1 - \tfrac{3}{4}\lambda\beta \operatorname{cotanh}^2(\beta/2) + O\left(\lambda^2\right)\right],$$

which coincides with the result (44.5).

In the second case we note quite generally (as a consequence of Baker–Hausdorf's formula)

$$\mathrm{e}^{x(a+a^\dagger)} = \mathrm{e}^{xa^\dagger}\,\mathrm{e}^{xa}\,\mathrm{e}^{x^2/2}.$$

It follows

$$\left(a+a^\dagger\right)^4 = a^{\dagger 4} + a^4 + 4a^\dagger a^3 + 4a^{\dagger 3} a + 6a + 6a^\dagger + 3 + 12 a^\dagger a + 6 a^{\dagger 2} a^2.$$

We now replace the quantum operators $a, a^\dagger$ by the classical variables $z, \bar{z}$. At this order only the terms with an equal number of a and $a^\dagger$ factors contribute. We then obtain

$$\operatorname{tr} \mathrm{e}^{-\beta H} = \frac{1}{2\sinh(\omega\beta/2)} \left[1 - \tfrac{3}{4}\lambda\beta\lambda\left(1 + 4\Delta(0) + 4\Delta^2(0)\right) + O\left(\lambda^2\right)\right].$$

We thus find that to recover the correct result we have to assign to $\theta(0)$ the value $\theta(0_+) = 0$.

Exercise 5.3

Consider, in the holomorphic representation of quantum mechanics, the states with wave function $\psi(z) = \mathrm{e}^{\alpha z}$, where α is an arbitrary complex number. Calculate the average of the position and momentum operators $\hat{q}, \hat{p}$ and then the dispersions $\Delta q = (\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2)^{1/2}$, $\Delta p = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{1/2}$. Show that the states ψ are minimal dispersion states: $\Delta q \Delta p = \frac{1}{2}$.

Solution. We first calculate the norm of $\psi(z)$ and find $\mathrm{e}^{\alpha\bar{\alpha}}$. We can then represent the position and momentum operators by

$$\hat{q} \;\mapsto\; (2\omega)^{-1/2}\left(z + \frac{\partial}{\partial z}\right), \qquad \hat{p} \;\mapsto\; i(\omega/2)^{1/2}\left(z - \frac{\partial}{\partial z}\right).$$

A short calculation then yields

$$\langle \hat{q} \rangle = (2/\omega)^{1/2} \operatorname{Re}\alpha\,, \quad \langle \hat{q} \rangle = (2\omega)^{1/2} \operatorname{Im}\alpha\,, \quad (\Delta q)^2 = \tfrac{1}{2\omega}\,, \quad (\Delta p)^2 = \tfrac{\omega}{2}\,,$$

which shows that $\psi(z)$ is a minimum dispersion wave function.

44.6 Quantum Evolution: from Particles to Fields : Chapter 6

Exercise 6.1

Use expression (6.24) to obtain a semiclassical approximation for the S-matrix of the one dimensional potential $V(q) = \lambda/\cosh^2 q$. Compare with exact results (to be found for example in R.G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York 1966)) for the forward and backward S-matrix elements, S_+ and S_- at energy $E = k^2/2$:

$$S_+ = \frac{\Gamma(d+\alpha)\Gamma(\alpha+1-d)}{\Gamma(1+\alpha)\Gamma(\alpha)}\,, \qquad S_- = \frac{\sin\pi d}{\sin\pi\alpha}\frac{\Gamma(d+\alpha)\Gamma(\alpha+1-d)}{\Gamma(1+\alpha)\Gamma(\alpha)}\,,$$

with

$$d = \tfrac{1}{2}(1-\sqrt{1-8\lambda}) \qquad \alpha = -ik\,, \quad k > 0\,.$$

.

Solution. We first briefly explain how the S-matrix can be calculated exactly. The corresponding Schrödinger equation is:

$$-\tfrac{1}{2}\psi'' + \frac{\lambda}{\cosh^2 x}\psi = \tfrac{1}{2}k^2\psi\,. \tag{44.13}$$

In terms of the function φ,

$$\psi(x) = \frac{1}{(2\cosh x)^\alpha}\varphi(x)\,, \qquad \alpha = -ik\,, \quad k > 0\,,$$

it becomes

$$-\varphi'' + 2\alpha\tanh x\varphi' + \frac{2\lambda+\alpha(\alpha+1)}{\cosh^2 x}\varphi = 0\,.$$

For later purpose it is useful to note

$$\frac{1}{(\mathrm{e}^x+\mathrm{e}^{-x})^\alpha} \sim \begin{cases} x\to-\infty & : \mathrm{e}^{\alpha x} \\ x\to+\infty & : \mathrm{e}^{-\alpha x} \end{cases}$$

We then set $t = \tanh x$ and find

$$(1-t^2)\varphi'' - 2(1+\alpha)\varphi' - \bigl(2\lambda+\alpha(\alpha+1)\bigr)\varphi = 0\,.$$

The solutions of this equation are hypergeometric functions, which in special cases degenerate into Jacobi's polynomials. The solution regular at $t=-1$ for example is

$$\varphi(t) = F\left(a,b;\tfrac{1}{2}(a+b+1);\tfrac{1}{2}(1+t)\right),$$

with a, b given by:

$$a = \alpha + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-8\lambda}\,, \qquad b = \alpha + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-8\lambda}\,.$$

For $\operatorname{Re}\tfrac{1}{2}(1+a+b) > \operatorname{Re} b > 0$, which implies $\lambda > 0$, the solution φ is given by Euler's formula

$$\varphi(t) = 2^b\int_1^\infty \left(s^2-1\right)^{(a-b-1)/2}(s-t)^{-a}\mathrm{d}s\,.$$

We find

$$\varphi(-1)=\frac{\Gamma(\alpha+1-d)\Gamma(d)}{\Gamma(1+\alpha)}\,,\qquad \text{with } d=\tfrac{1}{2}(a-b+1)=\tfrac{1}{2}\left(1-\sqrt{1-8\lambda}\right).$$

Using $a=d+\alpha$, we can write the behaviour of $\varphi(t)$ near $t=1$ as

$$\varphi(t)\underset{t\to1}{\sim}\frac{\Gamma(\alpha-d+1)\Gamma(-\alpha)}{\Gamma(1-d)}+2^{\alpha}\frac{\Gamma(d)\Gamma(\alpha)}{\Gamma(d+\alpha)}(1-t)^{-\alpha}.$$

The singular contribution corresponds to the incoming wave. Note that

$$(1-t)^{-\alpha}\sim 2^{-\alpha}\,\mathrm{e}^{2\alpha x}\,.$$

We conclude that the forward and backward S-matrix elements, S_+ and S_- are:

$$S_+=\frac{\Gamma(d+\alpha)\Gamma(\alpha+1-d)}{\Gamma(1+\alpha)\Gamma(\alpha)}\,,\qquad S_-=\frac{\sin\pi d}{\sin\pi\alpha}\frac{\Gamma(d+\alpha)\Gamma(\alpha+1-d)}{\Gamma(1+\alpha)\Gamma(\alpha)}\,.$$

These expressions, established for $\lambda>0$ have an analytic continuation for $\lambda<0$ and give the complete solution.

We can check the unitarity relation for d real and α imaginary

$$|S_+|^2+|S_-|^2=-\frac{\sin^2\pi\alpha}{\sin\pi(d-\alpha)\sin\pi(d+\alpha)}\left(1-\frac{\sin^2\pi d}{\sin^2\pi\alpha}\right)=1\,.$$

For $\lambda>1/8$ d is complex but the relation $d^*+d=1$ leads to the same identity.

Note that the potential is reflectionless for integer values of d, i.e. for $\lambda=-\frac{1}{2}n(n-1)$, situation in which the potential is attractive and the S-matrix a rational fraction of $\alpha=ik$

$$S(\alpha)=(-1)^{n-1}\prod_{m=1}^{n-1}\frac{m+\alpha}{m-\alpha}.$$

The S-matrix has $n-1$ poles corresponding to $n-1$ bound states.

Finally in the normalization of equation (44.6) we have the relation

$$n+\tfrac{1}{2}=E(1+g^2/4)^{-1/2}+\tfrac{1}{2}g\left(n^2+n+\tfrac{1}{2}\right),\quad \text{with } g^2=-1/2\lambda\,,$$

for the potential $\frac{1}{2}\tanh^2(q)$. This provides a check of the corresponding perturbative results.

Semiclassical limit. The semiclassical limit is obtained by letting k and λ becoming simultaneously large $k\propto\sqrt{\lambda}$. Then $d\sim-\sqrt{-2\lambda}$ and the Γ functions are replaced by their Stirling's approximation. Let us consider the different cases.

(i) $\lambda>0$, $\sigma=\sqrt{2\lambda}/k<1$ with the choice $d=-i\sqrt{2\lambda}$. We find:

$$\begin{aligned}\ln S_+&\sim-ik\left[(1+\sigma)\ln(1+\sigma)+(1-\sigma)\ln(1-\sigma)\right],\\ \ln S_-&\sim-ik\left[(1+\sigma)\ln(1+\sigma)+(1-\sigma)\ln(1-\sigma)\right]-\pi k(1-\sigma).\end{aligned}$$

We note that, as expected, the classically forbidden backward scattering is exponentially suppressed.

(ii) $\lambda > 0$, $\sigma = \sqrt{2\lambda}/k > 1$ with the choice $d = -i\sqrt{2\lambda}$. We find:

$$\begin{aligned}\ln S_+ &\sim -ik\left[(\sigma+1)\ln(\sigma+1) - (\sigma-1)\ln(\sigma-1)\right] - \pi k(\sigma-1),\\ \ln S_- &\sim -ik\left[(\sigma+1)\ln(\sigma+1) - (\sigma-1)\ln(\sigma-1)\right].\end{aligned}$$

We now note that the classically forbidden forward scattering is exponentially suppressed.

(iii) $\lambda < 0$, $\sigma = \sqrt{-2\lambda}/k$. We find:

$$\begin{aligned}\ln S_+ &\sim -ik\left[(1+i\sigma)\ln(1+i\sigma) + (1-i\sigma)\ln(1-i\sigma)\right],\\ \ln S_- &\sim -ik\left[(1+i\sigma)\ln(1+i\sigma) + (1-i\sigma)\ln(1-i\sigma)\right] - \pi k\,.\end{aligned}$$

The asymptotic form of S_- is only valid for d non integer. We note again that the classically forbidden backward scattering is exponentially suppressed.

Semiclassical calculation. We now perform the explicit calculation of the evolution operator using the path integral representation and the steepest descent method. We first consider the general hamiltonian

$$H = \tfrac{1}{2}p^2 + V(x),$$

where the potential is analytic and decreases fast enough for large arguments.

Forward scattering. We assume that the energy is larger than the maximal value of the potential so that classical forward scattering is possible. We then solve the equation of motion. Integrating once we obtain:

$$\tfrac{1}{2}\dot{x}^2(\tau) + V(x) = \tfrac{1}{2}\kappa^2,$$

with the boundary conditions

$$x(\tau') = x'\,, \quad x(\tau'') = x''\,.$$

We set $X = x'' - x'$, $T = \tau'' - \tau'$. We know from the general analysis of Section 6.1 that we are interested in trajectories which correspond to classical scattering, with $X/T = k$ finite when $\tau' \to -\infty$, $\tau'' \to \infty$. In these limits the boundary condition

$$T = \int_{x'}^{x''} \frac{\mathrm{d}x}{\sqrt{\kappa^2 - 2V(x)}},$$

leads to

$$\kappa = k + \frac{1}{T}\int_{-\infty}^{+\infty} \mathrm{d}x \left(\frac{k}{\sqrt{k^2 - 2V(x)}} - 1\right) + O\left(T^{-2}\right).$$

The action of the trajectory then is

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}\kappa^2 T - 2\int V(x)\mathrm{d}\tau = \tfrac{1}{2}\kappa^2 T - 2\int_{x'}^{x''} \frac{V(x)\mathrm{d}x}{\sqrt{\kappa^2 - 2V(x)}}\\ &= \tfrac{1}{2}k^2 T + \int_{-\infty}^{+\infty} \mathrm{d}x \left(\sqrt{k^2 - 2V(x)} - k\right) + O\left(T^{-1}\right).\end{aligned}$$

We then calculate the Fourier transform. The result depends only on $x''-x'$ and thus a factor of momentum conservation $\delta(k''-k')$ follows. The remaining integral over X is calculated by steepest descent. At leading order for T large we need only consider the terms of order T for the saddle point. We find

$$X = k'T \ \Rightarrow\ k' = k\,.$$

As shown in Section 6.1 the terms proportional to T cancel with the free motion factors, and we finally obtain the semiclassical result.

$$\ln S_+(k) = i\int_{-\infty}^{+\infty} \mathrm{d}x \left(\sqrt{k^2-2V(x)}-k\right).$$

In the special case of the hamiltonian (44.13) a short calculation yields for $\ln S_+(k)$ a result in agreement with the expansion of the exact result.

Backward scattering. We now assume that the energy is lower than the maximum value of the potential in such a way that a classical reflection is allowed. If we imagine starting from $-\infty$ we obtain after a similar calculation the classical action:

$$\mathcal{A} = \frac{1}{2T}\left(x'+x''-2x_0\right)^2 + 2\int_{-\infty}^{x_0} \mathrm{d}x \left(\sqrt{k^2-2V(x)}-k\right) + O\left(T^{-1}\right),$$

where x_0 is the reflection point $k^2 = 2V(x_0)$, and $k = (2x_0 - x' - x'')/T$.

Since the result only depends on the combination $x'+x''$ we find after Fourier transformation as expected a factor $\delta(k''+k')$. The remaining integral over $X = x'+x''$, calculated by steepest descent, yields:

$$2x_0 - X = k'T\,,\quad k' = k\,,$$

and thus

$$\ln S_-(k) = 2i\int_{-\infty}^{x_0} \mathrm{d}x \left(\sqrt{k^2-2V(x)}-k\right) + 2ikx_0\,.$$

The application to the hamiltonian (44.13) again shows agreement with the expansion of the exact expression. Note finally that the results in the forbidden region can be obtained by a proper analytic continuation in the potential.

Exercise 6.2

Following the ideas explained in Section 3.3, calculate the real time evolution operator e^{-iHT} in the case of the hamiltonian:

$$H = \tfrac{1}{2}p^2 + V(q), \quad \text{with} \quad \begin{cases} V = 0 & \text{for } q \in [0,\pi], \\ V = +\infty & \text{for } q \notin [0,\pi]. \end{cases}$$

Note that the result obtained by adding all the classical trajectories is incorrect and that a phase factor in front of half of the contributions is required. This difficulty reflects the singular nature of the function $V(q)$.

Solution. Let us first calculate the classical action of the various trajectories which satisfy the proper boundary conditions. It is possible to go from q' to q'' directly or after being reflected an arbitrary number of times by the two walls. One find two series

$$\mathcal{S}_\mathrm{c} = (q''-q'+2n\pi)^2/2T\,, \qquad \mathcal{S}_\mathrm{c} = (q''-q'+2n\pi)^2/2T\,.$$

After a shift of the classical trajectory the remaining gaussian integral is independent of the path and yields a constant normalization. If we would just add the two series we would find

$$U(q'',q';T)=(2\pi T)^{-1/2}\left(\sum_n \mathrm{e}^{i(q''-q'+2n\pi)^2/2T}+\mathrm{e}^{i(q''+q'+2n\pi)^2/2T}\right).$$

The sum can be evaluated, as in Section 3.3 using Poisson's formula and the result is

$$U(q'',q';\beta)=\sum_k \mathrm{e}^{iTk^2/2}\left(\mathrm{e}^{ik(q''-q')}+\mathrm{e}^{ik(q''+q')}\right). \tag{44.14}$$

This sum does not coincide with the exact result which can be written

$$U(q'',q';\beta)=\sum_k \mathrm{e}^{iTk^2/2}\left(\mathrm{e}^{ik(q''+q')}-\mathrm{e}^{ik(q''-q')}\right).$$

The eigenfunctions $\psi(q)$ corresponding to expression (44.14) are $\cos kq$ and thus satisfy the boundary conditions $\psi'|_{q=0,\pi}=0$ while the correct eigenfunctions satisfy $\psi|_{q=0,\pi}=0$. The correct result is obtained by multiplying the amplitude by a sign for each reflection. This phase factor which can only be obtained by a careful study of the limit of a smooth potential converging towards the square well using the semiclassical expressions of Section A39.2.

Exercise 6.3

Assume for the evolution operator a path-integral representation of the form

$$U(q'',q')=\int [\mathrm{d}q(t)\mathcal{N}(t)]\exp\left[\frac{i}{\hbar}\int \mathrm{d}t\,\mathcal{L}(q,\dot q)\right].$$

The term in the exponential then is fixed to be the lagrangian by demanding the correct classical limit. This leaves the normalization factor undetermined. Show by evaluating the evolution operator for t small that $\mathcal{N}(t)$ is fixed to have the form of Section 3.2 by unitarity. Note that one has to use the property that for t small only short trajectories contribute to the path-integral.

Solution. We explain the idea in the case of the lagrangian

$$\mathcal{L}(q,\dot q)=\tfrac{1}{2}g_{ij}(q)\dot q^i\dot q^j\,.$$

The short time evolution operator can then be written

$$U(q'',q')=\mathcal{N}(q'')\exp\left[\frac{i}{2\varepsilon}g_{ij}(q'')\left(q''^i-q'^i\right)\left(q''^j-q'^j\right)\right].$$

We then use the unitarity relation

$$\int \mathrm{d}q'\,U(q'',q')U^*(q,q')=\prod_i \delta(q^i-q''^i).$$

For ε small q' can be replaced by q or q'' in g_{ij} and $\mathcal{N}$. The remaining integral over q' then is gaussian. One finds

$$U(q'',q')U^*(q,q') = \mathcal{N}(q)\mathcal{N}(q'')\exp\left[\frac{i}{2\varepsilon}g_{ij}(q)\left(2q'^i - q^i - q''^i\right)\left(q''^j - q^j\right)\right].$$

The quadratic term in q' has cancelled, and the resulting integration yields a δ-function:

$$\int \mathrm{d}q'\, U(q'',q')U^*(q,q') = \mathcal{N}^2(q)\det{}^{-1}\big(g_{ij}(q)\big)\prod_i \delta(q^i - q''^i).$$

Unitarity thus implies

$$\mathcal{N}^2(q) = \det\big(g_{ij}(q)\big),$$

a result consistent with the continuation to real time of the expression (3.24) obtained by direct calculation.

Exercise 6.4

Verify the unitarity of the S-matrix (6.39).

Solution. It is convenient to introduce the notation

$$\sigma = \int \mathrm{d}t\ \mathrm{e}^{-i\omega t}\, b(t), \quad \bar\sigma = \int \mathrm{d}t\ \mathrm{e}^{i\omega t}\, \bar b(t).$$

The unitarity relation reads

$$\int \frac{\mathrm{d}z'\mathrm{d}\bar z'}{2i\pi}\bar{\mathcal{S}}(z',\bar z)\,\mathrm{e}^{-z'\bar z'}\,\mathcal{S}(z',\bar z) = \mathrm{e}^{z\bar z}\,.$$

We note

$$2\,\mathrm{Re}\left[\int_{-\infty}^{+\infty}\mathrm{d}t\mathrm{d}\tau\, b(t)\theta(t-\tau)\,\mathrm{e}^{-i\omega(t-\tau)}\,\bar b(\tau)\right] = \bar\sigma\sigma\,.$$

We then have to evaluate

$$K(\bar z,z) = \int \frac{\mathrm{d}z'\mathrm{d}\bar z'}{2i\pi}\exp\left[z\bar z' - i\left(\bar\sigma z + \sigma\bar z'\right) - \bar z' z' + i\left(\bar\sigma z' + \sigma\bar z\right) + \bar z z' - \bar\sigma\sigma\right].$$

This a simple gaussian integral which leads to the result.

44.7 Quantum Field Theory: Functional Methods : Chapter 7

Exercise 7.1

7.1.1. Write explicitly the field equation (7.32) for the generating functional of correlation functions in the case of the action

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x\left\{\tfrac{1}{2}\left[\partial_\mu\phi(x)\right]^2 + \tfrac{1}{2}m^2\phi^2(x) + \frac{g}{3!}\phi^3(x)\right\}.$$

Solution. The field equation reads:

$$\left[\left(-\nabla_x^2 + m^2\right)\frac{\delta}{\delta J(x)} + \frac{g}{2}\left(\frac{\delta}{\delta J(x)}\right)^2\right]\mathcal{Z}(J) = J(x)\mathcal{Z}(J).$$

7.1.2. Deduce the equations for correlation functions obtained by differentiating zero, one, two and three times with respect to the source.

Solution. Calling $Z^{(n)}$ the n-point correlation function we successively obtain

$$\begin{aligned}
&(-\nabla_x^2+m^2)Z^{(1)}(x)+\tfrac{1}{2}gZ^{(2)}(x,x)=0\,,\\
&(-\nabla_x^2+m^2)Z^{(2)}(x,y)+\tfrac{1}{2}gZ^{(3)}(x,x,y)=\delta(x-y),\\
&(-\nabla_x^2+m^2)Z^{(3)}(x,y_1,y_2)+\tfrac{1}{2}gZ^{(4)}(x,x,y_1,y_2)=\delta(x-y_1)Z^{(1)}(y_2)\\
&\quad+\delta(x-y_2)Z^{(1)}(y_1),\\
&(-\nabla_x^2+m^2)Z^{(4)}(x,y_1,y_2,y_3)+\tfrac{1}{2}gZ^{(5)}(x,x,y_1,y_2,y_3)=\delta(x-y_1)Z^{(2)}(y_2,y_3)\\
&\quad+\delta(x-y_2)Z^{(2)}(y_3,y_1)+\delta(x-y_2)Z^{(2)}(y_3,y_1).
\end{aligned}$$

7.1.3. Use these equations to calculate the one and three-point functions at order g and the two-point function at order g^2.

Solution. The first equation yields $Z^{(1)}(x)$ which by translation invariance is a constant from the two-point function at leading order. Denoting by $\Delta(x-y)$ the propagator we obtain

$$Z^{(1)}(x)=-\frac{g}{2m^2}\Delta(0).$$

From the last equation we obtain the four-point function at leading order, which is, as expected a disconnected sum of products of two-point functions (property of the gaussian measure).

$$Z^{(4)}(y_1,y_2,y_3,y_4)=\Delta(y_1-y_2)\Delta(y_3-y_4)+\ 2\text{ terms}.$$

We introduce these expressions in the equation for $Z^{(3)}$, and find

$$\begin{aligned}
Z^{(3)}(y_1,y_2,y_3)=-g\int &\mathrm{d}x\,\Delta(x-y_1)\Delta(x-y_2)\Delta(x-y_2)\\
&-\frac{g}{2m^2}\Delta(0)\left(\Delta(y_1-y_2)+\ 2\text{ terms}\right).
\end{aligned}$$

We can now calculate the two-point function

$$\begin{aligned}
Z^{(2)}(y_1-y_2)&=\Delta(y_1-y_2)-\tfrac{1}{2}g\int\mathrm{d}x\Delta(y_1-x)Z^{(3)}(x,x,y_2)\\
&=\Delta(y_1-y_2)+\tfrac{1}{2}g^2\int\mathrm{d}x\mathrm{d}z\,\Delta(y_1-x)\Delta^2(x-z)\Delta(z-y_2)\\
&\quad+\frac{g^2}{2m^2}\int\mathrm{d}x\,\Delta(y_1-x)\Delta(x-y_2)+\frac{g^2}{4m^4}\Delta^2(0)
\end{aligned}$$

Exercise 7.2

Write explicitly the field equations (7.32) for the various generating functionals in the example of the action:

$$\mathcal{S}(\phi)=\int\mathrm{d}^d x\left\{\tfrac{1}{2}\left[\partial_\mu\phi(x)\right]^2+\tfrac{1}{2}m^2\phi^2(x)+\frac{g}{3!}\phi^3(x)\right\},$$

and deduce from the equations the one-loop correction to the one, two and three 1PI correlation functions.

Solution. The quantum field equation reads:

$$\left[\left(-\nabla^2+m^2\right)\frac{\delta}{\delta J(x)}+\frac{g}{2}\left(\frac{\delta}{\delta J(x)}\right)^2\right]\mathcal{Z}(J)=J(x)\mathcal{Z}(J).$$

The generating functional of connected correlation functions then satisfies

$$\left[\left(-\nabla^2+m^2\right)\frac{\delta}{\delta J(x)}+\frac{g}{2}\left(\frac{\delta}{\delta J(x)}\right)^2\right]\mathcal{W}(J)+\frac{g}{2}\left(\frac{\delta\mathcal{W}}{\delta J(x)}\right)^2=J(x).$$

Finally for the 1PI functional we find

$$\frac{\delta\Gamma}{\delta\varphi(x)}=\left(-\nabla^2+m^2\right)\varphi(x)+\tfrac{1}{2}g\varphi^2(x)+\frac{g}{2}\left(\frac{\delta^2\Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)}\right)^{-1}\Bigg|_{x_1=x_2=x}. \tag{44.15}$$

If we omit the inverse two-point function we can integrate and obtain Γ in the tree approximation:

$$\Gamma(\varphi)=\mathcal{S}(\varphi).$$

Then iterating the equation we obtain the derivative of the one-loop functional:

$$\frac{\delta\Gamma_{1\,\mathrm{loop}}}{\delta\varphi(x)}=\frac{g}{2}\left[\left(-\nabla^2_{x_1}+m^2+g\varphi(x_1)\right)\delta(x_1-x_2)\right]^{-1}\Big|_{x_1=x_2=x}. \tag{44.16}$$

Expanding this expression in powers of g we find

$$\begin{aligned}\frac{\delta\Gamma_{1\,\mathrm{loop}}}{\delta\varphi(x)}&=\tfrac{1}{2}g\Delta(0)-\tfrac{1}{2}g^2\int\mathrm{d}y\,\Delta^2(x-y)\varphi(y)\\&\quad+\tfrac{1}{2}g^3\int\mathrm{d}y\mathrm{d}z\,\Delta(x-y)\varphi(y)\Delta(y-z)\varphi(z)\Delta(z-x)+O\left(g^4\right),\end{aligned}$$

where we have called Δ the propagator. Comparing then with the expansion of the 1PI functional we conclude

$$\begin{aligned}\Gamma_1^{(1)}&=\tfrac{1}{2}g\Delta(0)\\\Gamma_1^{(2)}&=-\tfrac{1}{2}g^2\Delta^2(x-y)\\\Gamma_1^{(3)}&=g^3\Delta(x-y)\Delta(y-z)\Delta(z-x).\end{aligned}$$

Note finally that in expression (44.16) we recognize the functional derivative of the expected tr ln:

$$g\left[\left(-\nabla^2+m^2++g\varphi(x_1)\right)\delta(x_1-x_2)\right]^{-1}\Big|_{x_1=x2=x}=\frac{\delta}{\delta\varphi(x)}\operatorname{tr}\ln\left[-\nabla^2+m^2+g\varphi(.)\right].$$

Exercise 7.3

Write the quantum field equations satisfied by the generating functionals of correlation functions in the example of the action:

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left(\tfrac{1}{2}\partial_\mu \phi(x)\partial_\mu \phi(x) + \tfrac{1}{2}m^2\phi^2(x) + \frac{1}{4!}g\phi^4(x) \right).$$

Recover the form of the 1PI functional at one-loop order.

Solution. The equation for the complete correlation functions has been given in Section 7.3. For the connected functions we obtain

$$\left[\left(-\nabla^2 + m^2\right) \frac{\delta}{\delta J(x)} + \frac{g}{6}\left(\frac{\delta}{\delta J(x)}\right)^3 \right] \mathcal{W}(J) + \frac{g}{6}\left(\frac{\delta \mathcal{W}}{\delta J(x)}\right)^3 + \frac{g}{2}\frac{\delta \mathcal{W}}{\delta J(x)}\frac{\delta^2 \mathcal{W}}{(\delta J(x))^2} = J(x).$$

Finally the 1PI functional satisfies

$$\begin{aligned}\frac{\delta \Gamma}{\delta \varphi(x)} = {} & \left(-\nabla^2 + m^2\right)\varphi(x) + \tfrac{1}{6}g\varphi^3(x) + \frac{g}{2}\varphi(x)\left(\frac{\delta^2 \Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)}\right)^{-1}\Bigg|_{x_1=x_2=x} \\ & + \tfrac{1}{6}g\mathcal{W}^{(3)}(J)\,,\end{aligned}$$

where the three-point connected function $\mathcal{W}^{(3)}(J)$ has the symbolic form

$$\mathcal{W}^{(3)}(J) \equiv \frac{\delta^3 \mathcal{W}(J)}{(\delta J)^3} = \Gamma^{(3)}(\varphi)\left(\Gamma^{(2)}(\varphi)\right)^{-1}\left(\Gamma^{(2)}(\varphi)\right)^{-1}\left(\Gamma^{(2)}(\varphi)\right)^{-1}.$$

One easily verifies that the last term does not contribute at one-loop order. Then, after integration, one finds in the tree approximation

$$\Gamma(\varphi) = \mathcal{S}(\varphi),$$

and thus at one-loop order

$$\frac{\delta \Gamma_{1\,\mathrm{loop}}}{\delta\varphi(x)} = \frac{g}{2}\varphi(x)\left(\frac{\delta^2 \mathcal{S}}{\delta\varphi(x_1)\delta\varphi(x_2)}\right)^{-1}\Bigg|_{x_1=x_2=x},$$

which, after integration, yields the general tr ln term

$$\Gamma_{1\,\mathrm{loop}} = \tfrac{1}{2}\operatorname{tr}\ln\frac{\delta^2 \mathcal{S}}{\delta\varphi\delta\varphi}.$$

44.8 Relativistic Fermions : Chapter 8

Exercise 8.1

8.1.1. Establish the quantum field equations in the case of the action (8.40) in the absence of ϕ-field self-interactions.

Solution. The field equations are

$$\left[\not{\partial} + M + g\frac{\delta}{\delta J(x)}\right]\frac{\delta}{\delta\bar\eta(x)}\mathcal{Z}(\bar\eta,\eta,J) = \eta(x)\mathcal{Z}(\bar\eta,\eta,J) \tag{44.17a}$$

$$\left[{}^T(-\not{\partial} + M) + g\frac{\delta}{\delta J(x)}\right]\frac{\delta}{\delta\eta(x)}\mathcal{Z}(\bar\eta,\eta,J) = \bar\eta(x)\mathcal{Z}(\bar\eta,\eta,J) \tag{44.17b}$$

$$\left[(-\nabla_x^2 + m^2)\frac{\delta}{\delta J(x)} + g\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar\eta(x)}\right]\mathcal{Z}(\bar\eta,\eta,J) = J(x)\mathcal{Z}(\bar\eta,\eta,J), \tag{44.17c}$$

where ^{T} means transposed.

8.1.2. Derive from these equations a set of relations between correlation functions and use them to determine the average of $\phi(x)$ at order g and the fermion and boson two-point functions at order g^2.

Solution. Setting $J = \eta = \bar\eta = 0$ in (44.17c) we immediately obtain the ϕ-field average:

$$(-\nabla_x^2 + m^2)\langle\phi(x)\rangle - g\left\langle\bar\psi(x)\psi(x)\right\rangle = 0\,,$$

and thus, because $\langle\phi(x)\rangle$ is x independent (translation invariance)

$$\langle\phi(x)\rangle = \frac{g}{m^2}\operatorname{tr}\Delta_{\mathrm{F}}(0).$$

If instead we first differentiate with respect to J we find

$$(-\nabla_x^2 + m^2)\langle\phi(x)\phi(y)\rangle - g\left\langle\phi(y)\bar\psi(x)\psi(x)\right\rangle = \delta(x-y).$$

Differentiating (44.17a) with respect to $\eta(y)$ yields

$$(\not{\partial}_x + M)_{\alpha\gamma}\left\langle\bar\psi_\beta(y)\psi_\gamma(x)\right\rangle + g\left\langle\phi(x)\bar\psi_\beta(y)\psi_\alpha(x)\right\rangle = \delta(x-y)\delta_{\alpha\beta}.$$

We also need equations (44.17c) differentiated with respect to η and $\bar\eta$, and (44.17a) with respect to $\bar\eta$ and twice η

$$\begin{aligned}
&(-\nabla_x^2 + m^2)\left\langle\phi(x)\bar\psi(y_1)\psi(y_2)\right\rangle - g\left\langle\bar\psi(y_1)\psi(y_2)\bar\psi(x)\psi(x)\right\rangle = 0\,,\\
&(\not{\partial}_x + M)\left\langle\bar\psi(y_3)\psi(x)\bar\psi(y_1)\psi(y_2)\right\rangle + g\left\langle\phi(x)\bar\psi(y_3)\psi(x)\bar\psi(y_1)\psi(y_2)\right\rangle\\
&\qquad\qquad = \delta(x-y_3)\left\langle\bar\psi(y_1)\psi(y_2)\right\rangle - \delta(x-y_1)\left\langle\bar\psi(y_3)\psi(y_2)\right\rangle\,.
\end{aligned}$$

In the last equation we set $g = 0$, replace the fermion two-point function by the propagator, and obtain the fermion four-point function at leading order (in the gaussian approximation). From the previous equation we then find the $\phi\bar\psi\psi$ vertex

$$\left\langle\phi(x)\bar\psi_\beta(z)\psi_\alpha(y)\right\rangle = \langle\phi\rangle\,(\Delta_{\mathrm{F}})_{\alpha\beta}\,(x-y) - g\int \mathrm{d}u\,\Delta_{\mathrm{B}}(x-u)\Delta_{\mathrm{F}}\left[\Delta_{\mathrm{F}}(y-u)\Delta_{\mathrm{F}}(u-z)\right]_{\alpha\beta}\,.$$

We then substitute in the equations for the two-point functions. The final solutions are:

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle &= \Delta_{\mathrm{B}}(x-y) + \langle\phi\rangle^2 - g^2\int \mathrm{d}z_1\,\mathrm{d}z_2\,\Delta_{\mathrm{B}}(x-z_1)\\
&\quad\times\operatorname{tr}\left[\Delta_{\mathrm{F}}(z_1-z_2)\Delta_{\mathrm{F}}(z_2-z_1)\right]\Delta_{\mathrm{B}}(z_2-y),\\
\left\langle\bar\psi(y)\psi(x)\right\rangle &= \Delta_{\mathrm{F}}(x-y)(1-g\langle\phi\rangle) + g^2\int \mathrm{d}z_1\,\mathrm{d}z_2\,\Delta_{\mathrm{F}}(x-z_1)\Delta_{\mathrm{F}}(z_1-z_2)\\
&\quad\times\Delta_{\mathrm{F}}(z_2-y)\Delta_{\mathrm{B}}(z_1-z_2),
\end{aligned}$$

where Δ_{B} and Δ_{F} are the boson and fermion propagators.

Exercise 8.2

Obtain the quantum field equations satisfied by the 1PI functional in the theory with bosons and fermions described by the action (8.40).

Solution. The field equations satisfied by the partition function are given in the preceding exercise. From them one derives

$$\begin{aligned} \frac{\delta \Gamma}{\delta \bar{\psi}(x)} &= [\not{\partial} + M + g\phi(x)]\,\psi(x) + g\mathcal{W}^{(2)}_{J\bar{\eta}}(x,x) \\ \frac{\delta \Gamma}{\delta \phi(x)} &= \left(-\nabla_x^2 + m^2\right)\phi(x) - g\bar{\psi}(x)\psi(x) + g\mathcal{W}^{(2)}_{\eta\bar{\eta}}(x,x), \end{aligned}$$

where we have defined

$$\mathcal{W}^{(2)}_{J\bar{\eta}}(x,y) = \frac{\delta^2 \mathcal{W}}{\delta J(x)\delta\bar{\eta}(y)}\,, \qquad \mathcal{W}^{(2)}_{\eta\bar{\eta}}(x,y) = \frac{\delta^2 \mathcal{W}}{\delta \eta(x)\delta\bar{\eta}(y)}\,.$$

The second functional derivatives of $\mathcal{W}$ form an operator and a 3×3 matrix in the $\{J, \eta, \bar{\eta}\}$ space, inverse of the analogous matrix for Γ. The explicit expressions in terms of Γ are too complicated to be reproduced here.

44.9 Quantum Field Theory: Divergences and Regularization : Chapter 9

Exercise 9.1

We consider the example of the $O(N)$ invariant action:

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}^d x \left(\tfrac{1}{2}\partial_\mu \boldsymbol{\phi}\partial_\mu \boldsymbol{\phi} + \tfrac{1}{2}m^2 \boldsymbol{\phi}^2 + \frac{1}{4!}\frac{g}{N}(\boldsymbol{\phi}^2)^2\right),$$

where $\boldsymbol{\phi}$ is a N-component vector.

9.1.1. Write the quantum field equation (see Chapter 7) satisfied by the generating functional of correlation functions.

Solution. The equation reads

$$\left[\left(-\nabla_x^2 + m^2\right)\frac{\delta}{\delta J_i(x)} + \frac{g}{6N}\frac{\delta}{\delta J_i(x)}\left(\frac{\delta}{\delta J_j(x)}\right)^2\right]\mathcal{Z}(J) = J_i(x)\mathcal{Z}(J).$$

9.1.2. Deduce a relation between the two and four-point functions.

Solution. Using the $O(N)$ invariance (assumed not spontaneously broken) we obtain

$$\left(-\nabla_x^2 + m^2\right) Z^{(2)}(x-y) + \frac{g}{6N^2} Z^{(4)}_{iijj}(y,x,x,x) = \delta(x-y).$$

9.1.3. As an approximation, assume now that the two correlation functions correspond to an unknown quadratic action (i.e. to a gaussian functional measure) and calculate the two-point function.

Solution. If the action is quadratic, the four-point function reduces to the sum of products of two-point functions:

$$Z^{(4)}_{ijkl}(x_1,x_2,x_3,x_4) = \delta_{ij}\delta_{kl} Z^{(2)}(x_1-x_2)Z^{(2)}(x_3-x_4) + \text{ 2 terms}\ .$$

Introducing this relation in the equation we find:

$$\left[\left(-\nabla_x^2 + m^2\right) + \frac{g(N+2)}{6N} Z^{(2)}(0)\right] Z^{(2)}(x-y) = \delta(x-y).$$

We set

$$\mu^2 = m^2 + \frac{g(N+2)}{6N} Z^{(2)}(0),$$

and thus after Fourier transformation

$$\tilde{Z}^{(2)}(p) = \frac{1}{p^2+\mu^2} \ \Rightarrow\ \mu^2 = m^2 + \frac{g(N+2)}{6N}\frac{1}{(2\pi)^d}\int^{\Lambda} \frac{\mathrm{d}^d p}{p^2+\mu^2}\,.$$

We note that this equation is meaningful only in the presence of a cut-off Λ for dimensions $d \ge 2$, which we indicate symbolically. It yields a relation between the "bare" mass m, the renormalized or physical mass μ and the cut-off. This relation becomes exact in the large N limit as the analysis of Chapter 30 shows.

Exercise 9.2

Repeat the whole analysis (quantum field equation and self-consistent free action approximation) for the so-called Gross–Neveu model (GN) discussed in Section 31.7, which is a model of N massless self-interacting Dirac fermions $\{\psi^i, \bar\psi^i\}$. The $U(N)$ symmetric action is:

$$\mathcal{S}\left(\bar{\boldsymbol\psi}, \boldsymbol\psi\right) = -\int \mathrm{d}^d x \left[\bar{\boldsymbol\psi}\cdot \not{\partial}\boldsymbol\psi + \tfrac{1}{2} g \left(\bar{\boldsymbol\psi}\cdot\boldsymbol\psi\right)^2\right].$$

The GN model has in even dimensions a discrete chiral symmetry:

$$\boldsymbol\psi \mapsto \gamma_S \boldsymbol\psi, \quad \bar{\boldsymbol\psi} \mapsto -\bar{\boldsymbol\psi}\gamma_S\,,$$

which forbids a fermion mass term.

Solution. Calling η and $\bar\eta$ the sources for $\bar\psi$ and ψ respectively we find:

$$\left[\not{\partial}_x \frac{\delta}{\delta\bar\eta_i(x)} + g\frac{\delta}{\delta\bar\eta_i(x)}\frac{\delta}{\delta\eta_k(x)}\frac{\delta}{\delta\bar\eta_k(x)}\right]\mathcal{Z}(\bar\eta,\eta) = \eta_i(x)\mathcal{Z}(\bar\eta,\eta).$$

A relation between the two and four-point functions follows by differentiating with respect to $\eta_j(y)$ and then setting the sources to zero:

$$\not{\partial}_x \left\langle \psi_i(x)\bar\psi_j(y)\right\rangle + g\left\langle \psi_i(x)\bar\psi_j(y)\psi_k(x)\bar\psi_k(x)\right\rangle = -\delta(x-y)\delta_{ij}\,.$$

Again assuming that the functions are calculated with a gaussian measure we can factorize the four-point function

$$\left\langle \psi(x)\bar\psi_j(y)\psi_k(x)\bar\psi_k(x)\right\rangle = \left\langle \psi_i(x)\bar\psi_j(y)\right\rangle\left\langle \psi_k(x)\bar\psi_k(x)\right\rangle - \left\langle \psi_k(x)\bar\psi_j(y)\right\rangle\left\langle \psi_i(x)\bar\psi_k(x)\right\rangle.$$

The function $<\psi(x)\bar\psi(x)>$ being a constant, we can solve the equation. After Fourier transformation, and taking into account $U(N)$ invariance, we find

$$\left\langle \psi_i(p)\bar\psi_j(-p)\right\rangle = \delta_{ij}\frac{1}{i\not{p}+\mu}\,,$$

where we have set

$$\mu = \frac{g(N-1)}{N}\left\langle \psi_k(x)\bar{\psi}_k(x)\right\rangle = g(N-1)\frac{\operatorname{tr}\mathbf{1}}{(2\pi)^d}\mu\int^{\Lambda}\frac{\mathrm{d}^d p}{p^2+\mu^2}\,.$$

We have here assumed that the regularization was chiral invariant. We see that this equation, called often the gap equation, has always the solution $\mu = 0$ which preserves the chiral symmetry. However, in $d > 2$ dimensions, when the coupling satisfies

$$g > g_c\,, \qquad g_c^{-1} = (N-1)\frac{\operatorname{tr}\mathbf{1}}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d p}{p^2}\,,$$

i.e. is sufficiently attractive, there exists another solution which breaks the chiral symmetry

$$\frac{1}{g} = \frac{(N-1)\operatorname{tr}\mathbf{1}}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d p}{p^2+\mu^2}\,,$$

and which, in addition, has a lower action. Within this approximation scheme, this model thus provides an example of non-perturbative spontaneous symmetry breaking and fermion mass generation, the Nambu–Jona-Lasinio mechanism, due to fermion pair condensation. Here again the approximation can be justified only in the large N limit (see Section 31.7).

44.10 Introduction to Renormalization Theory : Chapter 10

Exercise 10.1

Calculate $\Gamma^{(2)}_{\mathrm{r,as.}}$, as defined in Section 10.9.1, in an approximation in which only the term of highest degree in $\ln(p/m_{\mathrm{r}})$ is kept at each order (the leading log approximation). Discuss the result.

Solution. One verifies that the result is

$$\Gamma^{(2)}_{[\mathrm{r,as.}} = p^2 + C_1 p^2 \frac{g_{\mathrm{r}}}{1-\beta_2 g_{\mathrm{r}}\ln(p/m_{\mathrm{r}})}\,.$$

This expression has a pole when the momentum reaches the value

$$p^2 = m_{\mathrm{r}}^2\,\mathrm{e}^{2/\beta_2 g_{\mathrm{r}}},$$

(for $\beta_2 > 0$) often called the Landau pole. This singularity shows that even if g_{r} is small, perturbation theory will break at sufficiently high momentum. To go beyond this simple remark, more sophisticated RG arguments are required (see Chapter 25). Conversely if in some other theory $\beta_2 < 0$ then perturbation theory breaks down at sufficiently low momentum, indicating that the perturbative prediction of a massless spectrum is not reliable.

44.11 Dimensional Regularization and Minimal Subtraction : Chapter 11

Exercise 11.1

Characterize the form of renormalization constants in a minimal subtraction scheme with cut-off regularization, the parameter being Λ/m_{r} in which m_{r} is a renormalized mass scale (see Section 9.1.1).

Solution. Let us take the example of the coupling constant renormalization. Then the bare coupling constant g is related to the renormalized coupling constant by

$$g = g_{\mathrm{r}} + \ln \Lambda/m_{\mathrm{r}} G_2(g_{\mathrm{r}}) + (\ln \Lambda/m_{\mathrm{r}})^2 G_3(g_{\mathrm{r}}) + \cdots ,$$

where $G_n(g_{\mathrm{r}})$ is of order g_{r}^n. We can then calculate the β-function by differentiating with respect to m_{r} at g, Λ fixed

$$0 = \beta(g_{\mathrm{r}}) \left(1 + \sum_{n=1} (\ln \Lambda/m_{\mathrm{r}})^n G'_{n+1}(g_{\mathrm{r}})\right) - \sum_{n=0} (n+1)(\ln \Lambda/m_{\mathrm{r}})^n G_{n+2}(g_{\mathrm{r}}).$$

Since the β function is finite we conclude

$$\beta(g_{\mathrm{r}}) = G_2(g_{\mathrm{r}}), \qquad G_{n+2}(g_{\mathrm{r}}) = \frac{1}{n+1} G'_{n+1}(g_{\mathrm{r}}) \beta(g_{\mathrm{r}}).$$

In analogy with dimensional regularization all quantities are determined by the coefficient of $\ln(\Lambda/m_{\mathrm{r}})$.

Exercise 11.2

Pseudoscalar Yukawa interaction. Consider the action

$$\mathcal{S}\left(\phi, \psi, \bar{\psi}\right) = \int \mathrm{d}^d x \Big\{ -\bar{\psi}(x) \left[\not{\partial} + M + ig\gamma_S \phi(x)\right] \psi(x) + \tfrac{1}{2} \left(\partial_\mu \phi(x)\right)^2 + \tfrac{1}{2} m^2 \phi^2(x) + \frac{1}{4!} \lambda \phi^4(x) \Big\}, \tag{44.18}$$

in which ϕ is a pseudoscalar boson field and $\psi, \bar{\psi}$ Dirac fermions.

The factor i in front of γ_S is imposed by invariance under reflection hermiticity, i.e. euclidean hermitian conjugation followed by space reflection (see Section 8.2.1) which implies:

$$\gamma_1 \gamma_S^\dagger \left(ig\phi(x)\right)^* \gamma_1 = \gamma_S ig\phi(x).$$

In addition the action is reflection symmetric (see Section 8.2.2):

$$\begin{aligned} \psi(x) &\mapsto \gamma_1 \psi\left(\tilde{x}\right), \\ \bar{\psi}(x) &\mapsto \bar{\psi}\left(\tilde{x}\right) \gamma_1, \\ \phi(x) &\mapsto -\phi\left(\tilde{x}\right), \end{aligned}$$

with $\tilde{x} = (-x_1, x_2, \ldots, x_d)$.

The transformation of $\phi(x)$ is imposed by the factor γ_S in front of the $\phi\bar{\psi}\psi$ interaction and justifies the denomination pseudoscalar for the field $\phi\left(x\right)$.

Calculations can be done using dimensional regularization and minimal subtraction because it is never necessary to evaluate any trace of products of γ_S by an even number of γ_μ matrices.

11.2.1. Calculate the $\bar\psi\psi$ two-point function at one-loop order. It is suggested to expand on the basis $\{1, \not p\}$ and to use identities of the form

$$p \cdot q = \tfrac{1}{2}\left[q^2 + M^2 - (p-q)^2 - m^2\right] + \tfrac{1}{2}\left(p^2 + m^2 - M^2\right).$$

Solution. One finds in the notation of Section 11.7,

$$\begin{aligned} I_1(p) = {} & \left[\frac{N_d}{\varepsilon} - \frac{1}{16\pi^2}\ln\left(\frac{Mm}{\mu^2}\right)\right]\left(M + \frac{i\not p}{2}\right) - \frac{1}{32\pi^2}\frac{i\not p}{p^2}\left(M^2+m^2\right)\ln\left(\frac{M}{m}\right) \\ & - \frac{1}{16\pi^2}\left[M + \frac{i\not p}{2p^2}\left(p^2+m^2-M^2\right)\right]\left[\frac{1}{p^2}\sigma_+\sigma_-\ln\left(\frac{\sigma_+ + \sigma_-}{\sigma_+ - \sigma_-}\right)\right. \\ & \left. - 1 - \ln\left(\frac{M}{m}\right)\left(\frac{M^2-m^2}{p^2}\right)\right] + O(\varepsilon), \end{aligned} \qquad (44.19)$$

where we have set:

$$\sigma_\pm\left(p^2\right) = \sqrt{p^2 + (M \pm m)^2}. \qquad (44.20)$$

11.2.2. Calculate the divergent part of the ϕ two-point function at one-loop order.

Solution. The divergent part of the ϕ two-point function is:

$$I_2(p) = 4\left(M^2 + \frac{p^2}{2}\right)\frac{N_d}{\varepsilon} + O(1). \qquad (44.21)$$

11.2.3. Writing the renormalized action:

$$\begin{aligned} \mathcal{S}_{\mathrm{r}}\left(\phi, \psi, \bar\psi\right) = \int \mathrm{d}^d x \Big\{ & -Z_\psi\left[\bar\psi(x)\left(\not\partial + M_0 + i\gamma_S g_0 Z_\phi^{1/2}\,\phi(x)\right)\psi(x)\right] \\ & + \tfrac{1}{2}Z_\phi\left[\left(\partial_\mu\phi(x)\right)^2 + m_0^2\phi^2(x)\right] + \frac{1}{4!}\lambda_0 Z_\phi^2\phi^4(x)\Big\}, \end{aligned} \qquad (44.22)$$

calculate all renormalization constants and RG functions. One may introduce a renormalization scale μ and define the β-functions by

$$\beta_i(\lambda) = \mu\frac{\partial}{\partial\mu}\bigg|_{\lambda_0}\lambda_i\,. \qquad (44.23)$$

Compare with the results of Section 11.7 for $N = 1$.

Solution. We call M_0, m_0, g_0, λ_0 the bare parameters, M, m, g, λ the renormalized parameters, μ the renormalization scale.

Substituting in the Feynman diagrams and identifying the divergent parts we obtain equations for the renormalization constants:

$$\begin{aligned} Z_\psi\left(M_0 + i\not p\right) + N_d g^2\left(M + \frac{i\not p}{2}\right)\frac{1}{\varepsilon} &= \text{finite } O\left(2 \text{ loops}\right), \\ \mu^{-\varepsilon/2} g_0 Z_\psi Z_\phi^{1/2} - N_d\frac{g^3}{\varepsilon} &= \text{finite } O\left(2 \text{ loops}\right), \\ Z_\phi\left(p^2 + m_0^2\right) + N_d\left[-\frac{\lambda}{2}\frac{m^2}{\varepsilon} + 4\left(M^2 + \frac{p^2}{2}\right)\frac{g^2}{\varepsilon}\right] &= \text{finite } O\left(2 \text{ loops}\right), \\ \mu^{-\varepsilon}\lambda_0 Z_\phi^2 + N_d\left(-\frac{3}{2}\frac{\lambda^2}{\varepsilon} + \frac{24 g^4}{\varepsilon}\right) &= \text{finite } O\left(2 \text{ loops}\right). \end{aligned}$$

It follows that:

$$Z_\psi = 1 - N_d \frac{g^2}{2\varepsilon} + O\,(2 \text{ loops})\,, \tag{44.24}$$

$$Z_\phi = 1 - N_d \frac{2Ng^2}{\varepsilon} + O(2 \text{ loops}), \tag{44.25}$$

$$g_0 = m^{\varepsilon/2} g \left(1 + N_d \frac{(2N+3)}{2} \frac{g^2}{\varepsilon}\right) + O(2 \text{ loops}), \tag{44.26}$$

$$\lambda_0 = m^\varepsilon \left[\lambda + \frac{N_d}{\varepsilon}\left(\frac{3}{2}\lambda^2 + 4N\lambda g^2 - 24Ng^4\right)\right] + O(2 \text{ loops}), \tag{44.27}$$

$$m_0^2 = m^2 \left[1 + \frac{N_d}{\varepsilon}\left(\frac{\lambda^2}{2} + 2g^2\right)\right] + O(2 \text{ loops}). \tag{44.28}$$

The β-function. When bare and renormalized coupling constants are related by an equation of the form (11.108), the coupling constant RG functions $\beta_i\,(\lambda)$ are given by:

$$\beta_i\,(\lambda) = \mu \frac{\partial}{\partial_\mu}\bigg|_{\lambda_0} \lambda_i, \tag{44.29}$$

and thus using equation (11.108):

$$0 = \mu\, \frac{\partial}{\partial_\mu} \lambda_{0i}\bigg|_{\lambda_0} = \mu^\varepsilon \left(\varepsilon f_i\,(\lambda) + \beta_j\,(\lambda)\, \frac{\partial f_i}{\partial \lambda_j}\right). \tag{44.30}$$

A short calculation then yields:

$$\beta_{g^2} = -\varepsilon g^2 + N_d 5 g^4 + \cdots, \tag{44.31}$$

$$\beta_\lambda = -\varepsilon\lambda + N_d \left(\frac{3}{2}\lambda^2 + 4\lambda g^2 - 24 g^4\right) + \cdots. \tag{44.32}$$

11.2.4. Relate coupling and field renormalizations of models (11.106) and (44.18) by performing in action (44.18), restricted to $M = 0$, a chiral rotation on the fields $\psi, \bar\psi$ (see Section 8.2.3).

Solution. For $M = 0$ let us set

$$\psi = \mathrm{e}^{-i\pi\gamma_S/4}\,\psi'\,, \qquad \bar\psi = \bar\psi'\,\mathrm{e}^{-i\pi\gamma_S/4}\,.$$

Then the action becomes identical to the action of the model (11.106). This explains why the renormalization constants are the same.

A few additional results are:

$$M_0 = M\left(1 - \frac{1}{2} g^2 \frac{N_d}{\varepsilon}\right) + O(2 \text{ loops}), \tag{44.33}$$

$$m_0^2 = m^2 \left[1 + \left(\frac{1}{2}\lambda + 2g^2\right)\frac{N_d}{\varepsilon}\right] - 4M^2 g^2 \frac{N_d}{\varepsilon} + O(2 \text{ loops}). \tag{44.34}$$

44.12 Renormalization of Composite Operators : Chapter 12

Exercise 12.1

12.1.1. Study the renormalization of the ϕ^3 field theory in six dimensions,

$$\mathcal{S}(\phi)=\int \mathrm{d}^6 x\left[\tfrac{1}{2}\left(\partial_\mu \phi\right)^2+\tfrac{1}{2} m^2 \phi^2+\frac{1}{3!} g \phi^3\right],$$

and derive the renormalized quantum field equation.

Solution. Let us first note that, unlike in the ϕ^4 theory, the action is no longer reflection symmetric in ϕ and the ϕ has an non-vanishing expectation value. It is thus convenient to shift ϕ by a constant in such a way that $\langle\phi\rangle=0$. The action then takes the form

$$\mathcal{S}(\phi)=\int \mathrm{d}^6 x\left[\tfrac{1}{2}\left(\partial_\mu \phi\right)^2+\tfrac{1}{2} m^2 \phi^2+\frac{1}{3!} g \phi^3-c \phi\right].$$

In six dimensions the field ϕ has dimension two and the 1PI functions $\Gamma^{(l,n)}$ have dimension $6-2n-2l$. Power counting then tells us that the renormalized action has the form

$$\mathcal{S}_{\mathrm{r}}(\phi_{\mathrm{r}})=\int \mathrm{d}^6 x\left[\tfrac{1}{2} Z_\phi\left(\partial_\mu \phi_{\mathrm{r}}\right)^2+\tfrac{1}{2}(m_{\mathrm{r}}^2+\delta m^2) \phi_{\mathrm{r}}^2+\frac{1}{3!} g_{\mathrm{r}} Z_g \phi_{\mathrm{r}}^3-C \phi_{\mathrm{r}}\right].$$

The relations between bare and renormalized quantities are then

$$Z_\phi^{1/2} \phi_{\mathrm{r}}=\phi, \quad g=g_{\mathrm{r}} Z_g Z_\phi^{-3/2}, \quad m^2=(m_{\mathrm{r}}^2+\delta m^2) Z_\phi^{-1}, \quad C=c Z_\phi^{1/2}.$$

Possible renormalization conditions are $\langle\phi_{\mathrm{r}}(x)\rangle=0$ and

$$\begin{aligned}
\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p=0)&=m_{\mathrm{r}}^2, \\
\frac{\partial}{\partial p^2} \tilde{\Gamma}_{\mathrm{r}}^{(2)}(p)\big|_{p=0}&=1, \\
\tilde{\Gamma}_{\mathrm{r}}^{(3)}(0,0,0)&=g_{\mathrm{r}}.
\end{aligned}$$

The other primitively divergent functions are $\Gamma^{(1,1)}$, $\Gamma^{(1,2)}$ and $\Gamma^{(2,1)}$.

We now write the quantum field equation of the bare theory. We define the insertion of the ϕ^2 operator as the correlation function of $-\frac{1}{2}\phi^2$. We then find

$$(-\nabla^2+m^2) \frac{\delta \mathcal{Z}}{\delta J(x)}-g \mathcal{Z}_{\phi^2(x)}=\big(J(x)+c\big) \mathcal{Z}(J).$$

The equation for $\mathcal{W}(J)$ follows

$$(-\nabla^2+m^2) \frac{\delta \mathcal{W}}{\delta J(x)}-g \mathcal{W}_{\phi^2(x)}=J(x)+c.$$

After Legendre transformation we obtain

$$\frac{\delta \Gamma}{\delta \phi(x)}+c=(-\nabla^2+m^2) \phi(x)+g \Gamma_{\phi^2(x)}(\phi).$$

The equation for the renormalized 1PI functional then reads

$$\frac{\delta\Gamma_{\mathrm{r}}}{\delta\phi(x)}+C=Z_\phi^{1/2}\left[(-\nabla^2+m^2)Z_\phi^{1/2}\phi(x)+g_{\mathrm{r}}Z_gZ_\phi^{-3/2}\Gamma_{\phi^2(x)}(\phi Z_\phi^{1/2})\right].$$

This equation relates the renormalization constants due to the insertion of the $\phi^2(x)$ operator to the renormalization constants already present in the action: $\phi^2(x)$ is not really a new operator but is related to $\phi(x)$. Introducing the notation (in Fourier space)

$$\Gamma^{(n)}_{\phi^2}(q;p_1,\ldots,p_n)\equiv\Gamma^{(1,n)}(q;p_1,\ldots,p_n),$$

we can define renormalized correlation functions with one insertion as

$$\Gamma^{(1,n)}_{\mathrm{r}}(q;p_1,\ldots,p_n)=Z_gZ_\phi^{n/2-1}\Gamma^{(1,n)}(q;p_1,\ldots,p_n)\ \text{for}\ n>1\,,$$

which implies

$$\Gamma^{(n+1)}_{\mathrm{r}}(q,p_1,\ldots,p_n)=g_{\mathrm{r}}\Gamma^{(1,n)}_{\mathrm{r}}(q;p_1,\ldots,p_n).$$

In particular for $n=2$ we find a renormalization condition for the function $\Gamma^{(1,2)}$

$$\Gamma^{(1,2)}_{\mathrm{r}}(0,0,0)=1\,.$$

For $n=1$ instead we set

$$\Gamma^{(1,1)}_{\mathrm{r}}(p)=Z_gZ_\phi^{-1/2}\Gamma^{(1,1)}(p)+\left((Z_\phi-1)p^2+Z_\phi m^2-m_{\mathrm{r}}^2\right)/g_{\mathrm{r}}\,,$$

which corresponds to

$$\Gamma^{(2)}_{\mathrm{r}}(p)=p^2+m_{\mathrm{r}}^2+g_{\mathrm{r}}\Gamma^{(1,1)}_{\mathrm{r}}(p),$$

and the renormalization conditions

$$\Gamma^{(1,1)}_{\mathrm{r}}(p)=O\left(p^4\right).$$

12.1.2. Establish the corresponding Callan–Symanzik equations.

Solution. The relations between renormalized and bare functions lead to CS equations. Defining the operator

$$\mathrm{D}_{\mathrm{CS}}=m_{\mathrm{r}}\frac{\partial}{\partial m_{\mathrm{r}}}+\beta(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}}-\frac{n}{2}\eta(g_{\mathrm{r}})-l\eta_2(g_{\mathrm{r}}),$$

we find for $n>1$:

$$\mathrm{D}_{\mathrm{CS}}\Gamma^{(l,n)}_{\mathrm{r}}(q_1,\ldots,q_l;p_1,\ldots,p_n)=m_{\mathrm{r}}^2\sigma(g_{\mathrm{r}})\Gamma^{(l+1,n)}_{\mathrm{r}}(0,q_1,\ldots,q_l;p_1,\ldots,p_n),$$

valid except for $n=1, l=1,2$. Using renormalization conditions one immediately finds $\sigma(g_{\mathrm{r}})=2-\eta(g_{\mathrm{r}})$. Moreover, since we have related the renormalization of ϕ^2 to the renormalization of ϕ and g, the corresponding RG function $\eta_2(g_{\mathrm{r}})$ is related to $\eta(g_{\mathrm{r}})$ and $\beta(g_{\mathrm{r}})$

$$\eta_2(g_{\mathrm{r}})=\tfrac{1}{2}\eta(g_{\mathrm{r}})-\beta(g_{\mathrm{r}})/g_{\mathrm{r}}\,.$$

A simple algebra yields the CS equations for the remaining functions.

Exercise 12.2

In the ϕ^4_4 field theory, write explicitly the relation (12.33) in the case of dimensional regularization and minimal subtraction. Obtain two other equations by differentiating the functional integral with respect to the coupling constant and the mass, and determine the RG functions of the CS equation (12.15).

Solution. We start from the action (11.2) in the minimal subtraction scheme to establish the relation between dimension four operators. Following the method of Section 12.2 we find

$$-Z_\phi \Gamma_{\phi(x)\partial^2\phi(x)} + Z_2(g)m^2\Gamma_{\phi^2(x)} + \frac{1}{3!}gm^\varepsilon Z_g \Gamma_{\phi^4(x)} = \phi(x)\frac{\delta\Gamma}{\delta\phi(x)}. \tag{44.35}$$

By definition of $Z_2(g)$ the insertion of $Z_2\phi^2$, where we recall that ϕ is the renormalized field, is finite

$$[\phi^2(x)]_\mathrm{r} = Z_2(g)\phi^2(x).$$

Since the r.h.s. of equation (44.35) is finite we immediately conclude

$$[\phi(x)\partial^2\phi(x)]_\mathrm{r} = Z\phi\phi(x)\partial^2\phi(x) - \frac{1}{3!}gm^\varepsilon Z_g\phi^4(x).$$

Differentiating with respect to g we obtain

$$\frac{\partial\Gamma}{\partial g} = \tfrac{1}{2}Z'_\phi(g)\Gamma_{(\partial_\mu\phi)^2} + \tfrac{1}{2}m^2 Z'_2(g)\Gamma_{\phi^2} + \frac{1}{4!}m^\varepsilon\bigl(Z_g(g) + gZ'_g(g)\bigr)\Gamma_{\phi^4}\,. \tag{44.36}$$

The l.h.s. is finite. We conclude that the insertion at zero momentum of the sum of the operators in the r.h.s. is also finite. We can thus define

$$\begin{aligned}[\phi^4(x)]_\mathrm{r} = {}& \bigl(Z_g(g) + gZ'_g(g)\bigr)\phi^4(x) - 12m^{-\varepsilon}Z'_\phi(g)\phi(x)\partial^2\phi(x) \\ &+ 12m^{2-\varepsilon}Z'_2(g)\phi^2(x) + m^{-\varepsilon}A(g)\partial^2\phi^2(x).\end{aligned}$$

Note that we can also differentiate with respect to the mass:

$$m\frac{\partial\Gamma}{\partial m} = m^2 Z_2(g)\Gamma_{\phi^2} + \frac{1}{4!}\varepsilon m^\varepsilon gZ_g(g)\Gamma_{\phi^4}\,. \tag{44.37}$$

The equation (44.36) can be rewritten

$$\beta(g)\frac{\partial\Gamma}{\partial g} = \tfrac{1}{2}\eta(g)Z_\phi\Gamma_{(\partial_\mu\phi)^2} + \tfrac{1}{2}m^2(\eta(g)+\eta_2(g))Z_2(g)\Gamma_{\phi^2} + \frac{1}{4!}m^\varepsilon\bigl(2\eta(g)-\varepsilon\bigr)gZ_g\Gamma_{\phi^4}\,.$$

By integrating equation (44.35) over x and eliminating the dimension four operator insertions between (44.35,44.37,44.36) we recover the CS equations for correlation functions.

Moreover we can then directly calculate the elements of the 2×2 matrix which appears in the CS equations with the insertion of the two operators of dimension four. Indeed the renormalized $(\partial_\mu\phi)^2$ operator at zero momentum from (44.35) just corresponds $\int \mathrm{d}x\,\phi(x)\delta/\delta\phi(x)$, i.e. to the multiplication of $\Gamma^{(n)}$ by n, an operation which commutes with the CS differential operator. Thus the corresponding matrix elements are zero. For what concerns ϕ^4 at zero momentum, from (44.36) we see that it is equivalent to $m^{-\varepsilon}\partial/\partial g$. The commutation of this operator with the CS operator D_CS

$$D_\mathrm{CS} = m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \tfrac{1}{2}n\eta(g),$$

yields

$$[D_\mathrm{CS}, m^{-\varepsilon}\partial/\partial g] = -\varepsilon - \beta'(g) + \tfrac{1}{2}n\eta'(g).$$

The diagonal element is thus $-\varepsilon - \beta'(g)$ and the component on the operator $\phi\delta/\delta\phi = n$ is $\frac{1}{2}\eta'(g)$.

44.13 Symmetries and Renormalization : Chapter 13

An interesting exercise is to verify that the $SU(2) \times SU(2)$ linear σ-model with linear symmetry breaking can be renormalized at one-loop order consistently with WT identities. Some help can be found in the following references

B.W. Lee, *Chiral Dynamics*, Gordon and Breach, New-York 1972;

J.L. Basdevant and B.W. Lee, *Phys. Rev.* D2 (1970) 1680;

and the proceedings of the Cargèse summer institute 1970, D. Bessis ed. Cargèse Lectures in Physics vol. 5, Gordon and Breach, New-York 1972.

We propose here instead to study the structure of symmetry breaking, in the classical limit, for an effective $SU(3) \times SU(3)$ invariant renormalizable field theory. The first reference is also relevant for this exercise. We consider as fundamental fields the scalar and pseudoscalar (π, K, η, η') bound states of the three $\mathbf{u}, \mathbf{d}, \mathbf{s}$ quarks. They can be described by a 3×3 complex matrix $\mathbf{M}$ which transforms under $SU(3) \times SU(3)$ as:

$$\mathbf{M}(x) \mapsto \mathbf{U}_1^\dagger \mathbf{M}(x) \mathbf{U}_2 \,, \quad \mathbf{U}_1 \,, \mathbf{U}_2 \in SU(3) \,.$$

Exercise 13.1

Write the most general renormalizable $SU(3) \times SU(3)$ invariant scalar action, in four dimensions, and discuss the general pattern of spontaneous symmetry breaking.

Solution: The action can be written

$$\begin{aligned} \mathcal{S}(\mathbf{M}) = \int \mathrm{d}^4 x \Big[& \mathrm{tr}\left(\partial_\mu \mathbf{M}^\dagger \partial_\mu \mathbf{M} + m^2 \mathbf{M}^\dagger \mathbf{M} + \tfrac{1}{2} h \left(\mathbf{M}^\dagger \mathbf{M}\right)^2\right) + \tfrac{1}{2} g \left(\mathrm{tr}\, \mathbf{M}^\dagger \mathbf{M}\right)^2 \\ & - u \left(\det \mathbf{M}^\dagger + \det \mathbf{M}\right)\Big] . \end{aligned} \tag{44.38}$$

An interaction proportional to $\det \mathbf{M}$ is allowed because for a 3×3 matrix it is equivalent to a cubic interaction, owing to the identity:

$$6 \det \mathbf{M} = (\mathrm{tr}\, \mathbf{M})^3 + 2\, \mathrm{tr}\, \mathbf{M}^3 - 3\, \mathrm{tr}\, \mathbf{M}\, \mathrm{tr}\, \mathbf{M}^2 .$$

Note that for the last two terms in (44.38) we have made a conventional choice of global phase for $\mathbf{M}$. When u (which we can choose positive) vanishes the symmetry group becomes instead $U(3) \times U(3)$ (remember the problem of the $U(1)$ anomaly). Finally since

$$\mathrm{tr}\left(\mathbf{M}^\dagger \mathbf{M}\right)^2 \le \left(\mathrm{tr}\, \mathbf{M}^\dagger \mathbf{M}\right)^2 \le 3\, \mathrm{tr}\left(\mathbf{M}^\dagger \mathbf{M}\right)^2 ,$$

we have the restrictions or g, h positive or if one is negative

$$h - 3|g| \ge 0 \,, \qquad \text{or} \quad g - |h| \ge 0 \,.$$

Spontaneous symmetry breaking. A general complex 3×3 matrix $\mathbf{M}$ can be written

$$\mathbf{M} = \mathrm{e}^{i\theta/3}\, \mathbf{U}_1^\dagger \Lambda \mathbf{U}_2 \,,$$

where U_1 and U_2 are two $SU(3)$ matrices, and Λ is diagonal with positive matrix elements λ_i. Only the determinants depend on the angle θ:

$$\det \mathbf{M}^\dagger + \det \mathbf{M} = 2 \cos\theta \det \Lambda \,.$$

For $u > 0$ the minimum of the corresponding term is at $\theta = 0$. Minimizing the potential with respect to λ_i we then get

$$\left(m^2 + g \operatorname{tr} \Lambda^2\right) \lambda_i^2 + h\lambda_i^4 - u \det \Lambda = 0 \ \text{ or } \ \left(m^2 + g \operatorname{tr} \Lambda^2\right) \Lambda^2 + h\Lambda^4 - u \det \Lambda = 0\,. \quad (44.39)$$

We now assume that the parameters are such that the equation has non-trivial roots. We then find two cases

(i) $h > 0$: At $\det \Lambda, \operatorname{tr} \Lambda^2$ fixed, the equation has only one positive root in λ_i^2, and since $\lambda_i > 0$ all three λ_i are equal, $\lambda_i = \lambda$, the matrix Λ is proportional to the identity and the residual symmetry is $SU(3)$:

$$\mathbf{U}_1^\dagger \lambda \mathbf{1} \mathbf{U}_2 = \lambda \mathbf{1} \ \Rightarrow \ \mathbf{U}_1 = \mathbf{U}_2\,.$$

Equation (44.39) becomes:

$$(3g + h)\lambda^2 - u\lambda + m^2 = 0\,.$$

We know $3g + h \geq 0$ and thus the equation has positive solutions either if $m^2 < 0$ or if $u^2 - 4m^2(3g + h) \geq 0$.

(ii) $h < 0$: Again at $\det \Lambda, \operatorname{tr} \Lambda^2$ fixed, if the equation has one positive root in λ_i^2, it has two, and the λ_i can take two different values. Let us call λ the value which appears twice λ' the other one. Thus a new pattern of SSB is possible where $SU(3) \times SU(3)$ is broken down generically to $U(2)$:

$$\mathbf{U}_1^\dagger \Lambda \mathbf{U}_2 = \Lambda \text{ with } \Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda' \end{pmatrix}$$
$$\Rightarrow \ \mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U} \text{ with } U_{i3} = U_{3i} = 0 \text{ for } i = 1, 2\,.$$

Expressing that $\lambda \neq \lambda'$ we derive from equation (44.39) a simple relation

$$h\lambda' + u = 0\,. \quad (44.40)$$

(iii) There is an exceptional case when the double value λ vanishes which leads to $SU(2) \times SU(2) \times U(1)$. Equation (44.39) then reduces to

$$(g + h)\lambda'^2 + m^2 = 0\,.$$

Exercise 13.2

Identify the Goldstone modes in the different cases first by purely group theory considerations and then by determining the mass matrix.

Solution:

(i) $SU(3)$ has 8 generators. In case (i) the number of Goldstone particles is thus $8 + 8 - 8 = 8$. To identify them we consider infinitesimal $SU(3)$ transformations which do not belong to the $SU(3)$ subgroup. We can parametrize them in terms of a traceless hermitian matrix $\mathbf{H}$:

$$\mathbf{U}_1 = \mathbf{1} - i\tfrac{1}{2}\mathbf{H}\,, \qquad \mathbf{U}_2 = \mathbf{1} + i\tfrac{1}{2}\mathbf{H}\,.$$

Acting on the vacuum expectation value Λ we get $i\lambda\mathbf{H}$, i.e. a traceless antihermitian matrix. The conclusion are that the massless states correspond to an octet of pseudoscalar particles. There are three types of massive states, one scalar and one pseudoscalar singlet, one scalar octet.

Remark. Note that in the special case $u = 0$ the symmetry group becomes $SU(3) \times SU(3) \times U(1)$ and the same vacuum expectation value leads to a nonet of Goldstone pseudoparticles. Since such a symmetry group is generated in QCD by assuming the quarks $\mathbf{u}, \mathbf{d}, \mathbf{s}$ massless, this leads to the famous $U(1)$ problem.

Let us now consider the mass matrix corresponding to action (44.38). We set $\mathbf{M} = \lambda + (\Sigma + i\Pi)/\sqrt{2}$ and expand the different terms up quadratic order in the fields:

$$\begin{aligned}
\operatorname{tr}\mathbf{M}^\dagger\mathbf{M} - \operatorname{tr}\Lambda^2 &= \operatorname{tr}\left(\sqrt{2}\lambda\Sigma + \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}\Sigma^2\right)\\
\operatorname{tr}\left(\mathbf{M}^\dagger\mathbf{M}\right)^2 - \operatorname{tr}\Lambda^4 &= \lambda^2\operatorname{tr}\left(2\sqrt{2}\lambda\Sigma + \Pi^2 + 3\Sigma^2\right) + \cdots\\
\left(\operatorname{tr}\mathbf{M}^\dagger\mathbf{M}\right)^2 - \left(\operatorname{tr}\Lambda^2\right)^2 &= \lambda^2\operatorname{tr}\left(6\sqrt{2}\lambda\Sigma + 3\Pi^2 + 3\Sigma^2\right) + 2\lambda^2\left(\operatorname{tr}\Sigma\right)^2 + \cdots\\
\det\mathbf{M}^\dagger + \det\mathbf{M} - 2\det\Lambda &= \tfrac{1}{2}\lambda\left[\operatorname{tr}\left(2\sqrt{2}\lambda\Sigma + \Pi^2 - \Sigma^2\right) - \left(\operatorname{tr}\Pi\right)^2 + \left(\operatorname{tr}\Sigma\right)^2\right] + \cdots,
\end{aligned}$$

where we have used for the last equation $\ln\det = \operatorname{tr}\ln$. By definition of λ the coefficient of $\operatorname{tr}\Sigma$ must vanish. We then observe that $\operatorname{tr}\Pi^2$ only appears in the combination $2\sqrt{2}\lambda\Sigma + \Pi^2$. Thus the coefficient of $\operatorname{tr}\Pi^2$ also vanishes. The only remaining dependence in Π is through $\left(\operatorname{tr}\Pi\right)^2$. Thus, as anticipated, the determinants give a mass to the $SU(3)$ pseudoscalar singlet. Finally we note that the scalar fields are all massive but with a different mass for the singlet and octet. Eliminating the parameter m^2 with the help of equation (44.39) we find

$$m^2_{\Pi_1} = 3u\lambda \quad m^2_{\Sigma_1} = (6g + 2h)\lambda^2 - u\lambda\,, \quad m^2_{\Sigma_8} = 2h\lambda^2 + 2u\lambda\,.$$

(ii) We now expect $8 + 8 - 4 = 12$ massless Goldstone particles. To identify them we now set

$$\mathbf{U}_1 = \mathbf{1} - i\tfrac{1}{2}\mathbf{H}_1\,, \quad \mathbf{U}_2 = \mathbf{1} + i\tfrac{1}{2}\mathbf{H}_2\,,$$

where $\mathbf{H}_1, \mathbf{H}_2$ are traceless hermitian matrices, and impose

$$\left(\mathbf{H}_1\right)_{ij} = \left(\mathbf{H}_2\right)_{ij} \quad \text{for } i,j \in \{1,2\} \text{ and } i = j = 3\,.$$

Acting on Λ we find that the Goldstone particles form a singlet, two doublets and a triplet of $SU(2)$ for what concerns the pseudoscalars, and two doublets of scalars. Again we verify these properties on the mass matrix. We set $\mathbf{M} = \lambda + (\Sigma + i\Pi)/\sqrt{2}$ and expand the different terms up to quadratic order in the fields:

$$\begin{aligned}
\operatorname{tr}\mathbf{M}^\dagger\mathbf{M} - \operatorname{tr}\Lambda^2 &= \operatorname{tr}\left(\sqrt{2}\Lambda\Sigma + \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}\Sigma^2\right)\\
\operatorname{tr}\left(\mathbf{M}^\dagger\mathbf{M}\right)^2 - \operatorname{tr}\Lambda^4 &= \operatorname{tr}\left(2\sqrt{2}\Lambda^3\Sigma + 2\Lambda^2\Pi^2 - (\Lambda\Pi)^2 + 2\Lambda^2\Sigma^2 + (\Lambda\Sigma)^2\right) + \cdots\\
\left(\operatorname{tr}\mathbf{M}^\dagger\mathbf{M}\right)^2 - \left(\operatorname{tr}\Lambda^2\right)^2 &= \operatorname{tr}\Lambda^2\operatorname{tr}\left(2\sqrt{2}\Lambda\Sigma + \Pi^2 + \Sigma^2\right) + 2\left(\operatorname{tr}\Lambda\Sigma\right)^2 + \cdots\\
\det\mathbf{M}^\dagger + \det\mathbf{M} - 2\det\Lambda &= \tfrac{1}{2}\det\Lambda\left[\operatorname{tr}\left(2\sqrt{2}\Lambda^{-1}\Sigma + \left(\Lambda^{-1}\Pi\right)^2 - \left(\Lambda^{-1}\Sigma\right)^2\right)\right.\\
&\qquad \left. - \left(\operatorname{tr}\Lambda^{-1}\Pi\right)^2 + \left(\operatorname{tr}\Lambda^{-1}\Sigma\right)^2\right] + \cdots\,.
\end{aligned}$$

Using eqs. (44.39,44.40), one verifies that all terms depending on the Π field cancel except the term proportional to $u(\operatorname{tr}\Lambda^{-1}\Pi)^2$ which corresponds to a $SU(2)$ singlet. In the same way one checks that the coefficients of the terms $|\Sigma_{i3}|^2$, $i=1,2$, vanish.

(iii) The last case can be treated in a completely similar way. The 9 Goldstone bosons are 5 pseudoscalars, a singlet and a $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of $SU(2)\times SU(2)$ and 4 scalars in the same representation.

Exercise 13.3

Spontaneous symmetry breaking can lead to a realistic residual symmetry but not of course to realistic masses. In particular the explicit $SU(3)$ symmetry breaking is rather large in the scale of physics we want to describe. The explicit breaking of $SU(2)\times SU(2)$ is smaller but responsible for the pion mass. Therefore to make the model more realistic we add a linear breaking term:

$$\mathcal{S}_B = \mathcal{S}_{\text{sym.}} - \operatorname{tr}\mathbf{C}\left(\mathbf{M}+\mathbf{M}^\dagger\right),$$

where $\mathbf{C}$ is a diagonal matrix with diagonal elements c, c, c'. The model then depends on 6 parameters. Determine analytically the mass spectrum. Is it possible to adjust the parameters in such a way that the pseudoscalar decay constants and the masses agree in a reasonable way with the experimental values? This last exercise requires a particle data table and a calculator.

Solution. With the linear symmetry breaking term, the residual symmetry is the diagonal $SU(2)$ subgroup of case (ii) of exercise 13.2. The matrix $\mathbf{C}$ is a diagonal matrix similar to Λ. Its addition allows to use λ, λ' as free parameters, related to $\mathbf{C}$ by:

$$\mathbf{C} = \left(m^2 + g\operatorname{tr}\Lambda^2\right)\Lambda + h\Lambda^3 - u\det\Lambda\Lambda^{-1}.$$

One finds for the pseudoscalar particles the following analytic expressions:

$$\begin{aligned}
m^2_{\Pi_3} &\equiv m^2_\pi = m^2 + g\left(2\lambda^2+\lambda'^2\right) + h\lambda^2 - u\lambda',\\
m^2_{\Pi_2} &\equiv m^2_K = m^2_\pi + (u+h\lambda')(\lambda'-\lambda),\\
m^2_\eta &= m^2_\pi + \tfrac{1}{2}h\left(\lambda'^2-\lambda^2\right) + \tfrac{3}{2}u\lambda' - \tfrac{1}{2}\left[\left(h\left(\lambda^2-\lambda'^2\right)+u\lambda'\right)^2 + 8u^2\lambda^2\right]^{1/2},\\
m^2_{\eta'} &= m^2_\pi + \tfrac{1}{2}h\left(\lambda'^2-\lambda^2\right) + \tfrac{3}{2}u\lambda' + \tfrac{1}{2}\left[\left(h\left(\lambda^2-\lambda'^2\right)+u\lambda'\right)^2 + 8u^2\lambda^2\right]^{1/2}.
\end{aligned}$$

Again in the limit $u\to 0$ one finds results inconsistent with experimental data since then $m_\eta = m_\pi$ and $m_K \sim m_{\eta'}$.

For the scalars the masses are

$$\begin{aligned}
m^2_{\Sigma_3} &= m^2_\pi + 2h\lambda^2 + 2u\lambda',\\
m^2_{\Sigma_2} &\equiv m^2_\kappa = m^2_\pi + (u+h\lambda')(\lambda+\lambda'),\\
m^2_{\Sigma_1\pm} &= m^2_\pi + \tfrac{1}{2}h\left(\lambda^2+3\lambda'^2\right) + g\left(2\lambda^2+\lambda'^2\right) + \tfrac{1}{2}u\lambda' g \pm \tfrac{1}{2}\sqrt{(A-C)^2+4B^2},
\end{aligned}$$

with we have defined

$$\begin{aligned}
A &= (3h+4g)\lambda^2 - u\lambda'\\
B &= \sqrt{2}\lambda(2g\lambda'-u)\\
C &= (3h+2g)\lambda'^2.
\end{aligned}$$

Consistency with experimental masses requires

$$\lambda' > \lambda\,, \qquad -\frac{1}{\lambda'} < \frac{h}{u} < \frac{1}{\lambda' - \lambda}\,.$$

If we determine the parameters by imposing the physical values of the pseudoscalar masses we find the solution

$$\lambda'/\lambda = 1.80..., \quad u\lambda/m_\pi = 10.4..., \quad h\lambda^2/m_\pi^2 = 2.42...\,.$$

The last coupling constant only appears in the scalar masses and is thus poorly determined $g\lambda^2/m_\pi^2 \sim 10$. Finally the matrix $\mathbf{C}$ has on its diagonal $\lambda m_\pi^2, \lambda m_\pi^2, (\lambda+\lambda')m_K^2 - \lambda m_\pi^2$.

44.14 The Non-Linear σ-Model: An Example of Non-Linearly Realized Symmetries : Chapter 14

Exercise 14.1

Verify explicitly that when the action $\mathcal{S}(\pi, H)$ is $O(N-1)$ invariant and satisfies the relation (14.35) then the differential operators $\mathcal{D}_i$

$$\mathcal{D}_i = \int \mathrm{d}x \left[\frac{\delta\mathcal{S}}{\delta\pi_i(x)}\frac{\delta}{\delta H(x)} + \frac{\delta\mathcal{S}}{\delta H(x)}\frac{\delta}{\delta\pi_i(x)}\right],$$

and

$$\mathcal{D}_{ij} = \int \mathrm{d}x \left[\pi_i(x)\frac{\delta}{\delta\pi_j(x)} - \pi_j(x)\frac{\delta}{\delta\pi_i(x)}\right],$$

form a representation of the Lie algebra of $O(N)$.

Solution: One first verifies that the space dependence plays no role and can thus be omitted. Then

$$[\mathcal{D}_i, \mathcal{D}_j] = \left[\frac{\partial\mathcal{S}}{\partial\pi_i}\frac{\partial^2\mathcal{S}}{\partial\pi_j\partial H} + \frac{\partial\mathcal{S}}{\partial H}\frac{\partial^2\mathcal{S}}{\partial\pi_i\partial\pi_j}\right]\frac{\partial}{\partial H} + \left[\frac{\partial\mathcal{S}}{\partial\pi_i}\frac{\partial^2\mathcal{S}}{\partial H\partial H} + \frac{\partial\mathcal{S}}{\partial H}\frac{\partial^2\mathcal{S}}{\partial\pi_i\partial H}\right]\frac{\partial}{\partial\pi_j} - (i \leftrightarrow j).$$

Differentiating equation (14.35) with respect to π_j and H respectively we obtain

$$\frac{\partial\mathcal{S}}{\partial\pi_i}\frac{\partial^2\mathcal{S}}{\partial\pi_j\partial H} + \frac{\partial\mathcal{S}}{\partial H}\frac{\partial^2\mathcal{S}}{\partial\pi_i\partial\pi_j} + H\delta_{ij} = 0\,,$$
$$\frac{\partial\mathcal{S}}{\partial\pi_i}\frac{\partial^2\mathcal{S}}{\partial H\partial H} + \frac{\partial\mathcal{S}}{\partial H}\frac{\partial^2\mathcal{S}}{\partial\pi_i\partial H} + \pi_i = 0\,.$$

It follows

$$[\mathcal{D}_i, \mathcal{D}_j] = -\mathcal{D}_{ij}\,.$$

To calculate the commutator $[\mathcal{D}_{ij}, \mathcal{D}_k]$ we note that

$$\mathcal{D}_{ij}\mathcal{S} = 0\,, \qquad \mathcal{D}_{ij}\frac{\partial\mathcal{S}}{\partial H} = 0\,,$$

as a consequence of the $O(N-1)$ invariance of the action. Also

$$\mathcal{D}_{ij}\frac{\partial\mathcal{S}}{\partial\pi_k} - \frac{\partial}{\partial\pi_k}\mathcal{D}_{ij}\mathcal{S} = -\delta_{ik}\frac{\partial\mathcal{S}}{\partial\pi_j} + \delta_{jk}\frac{\partial\mathcal{S}}{\partial\pi_i}.$$

Therefore

$$[\mathcal{D}_{ij}, \mathcal{D}_k] = \left[\delta_{jk}\frac{\partial\mathcal{S}}{\partial\pi_i} - \delta_{ik}\frac{\partial\mathcal{S}}{\partial\pi_j}\right]\frac{\partial}{\partial H} + \frac{\partial\mathcal{S}}{\partial H}\left(\delta_{jk}\frac{\partial}{\partial\pi_i} - \delta_{ik}\frac{\partial}{\partial\pi_j}\right) = \delta_{jk}\mathcal{D}_i - \delta_{ik}\mathcal{D}_j\,.$$

Exercise 14.2

We consider the partition function of the $O(4) \sim SU(2) \times SU(2)$ non-linear σ-model

$$\mathcal{Z} = \int \left[\left(f^2 - \pi^2(x) \right)^{-1/2} \mathrm{d}\pi(x) \right] \exp\left[-\mathcal{S}(\pi, c) \right], \tag{44.41}$$

with the action

$$\mathcal{S}(\pi, c) = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left[(\partial_\mu \pi(x))^2 + \frac{(\pi \cdot \partial_\mu \pi(x))^2}{f^2 - \pi^2(x)} \right] - c\sqrt{f^2 - \pi^2(x)} \right\}. \tag{44.42}$$

It what follows it will be convenient to set $c = \mu^2 f$.

We propose to study this model at one-loop order, to verify in particular that it can be renormalized with only two renormalization constants Z, Z_f as proven in Chapter 14.

It will be convenient to set $d = 2 + \varepsilon$ and to define a dimensionless coupling constant g:

$$f^{-2}\Gamma(1 - \varepsilon/2)\mu^\varepsilon (4\pi)^{-\varepsilon/2} = g\,.$$

Dimensional regularization will everywhere be used.

14.2.1. Establish the Feynman rules. For the calculations we have in mind only the propagator, the four-point and six-point vertices are required.

Solution:

$$\begin{aligned}
\Delta_2(p) &= \frac{1}{p^2 + \mu^2}\,, \\
V^{(4)} &= \frac{1}{8f^2} \delta_{i_1 i_2} \delta_{i_3 i_4} \left((p_1 + p_2)^2 + \mu^2 \right), \\
V^{(6)} &= \frac{1}{16 f^4} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \left((p_1 + p_2)^2 + \mu^2 \right).
\end{aligned}$$

14.2.2. Write the renormalized action.

Solution: The renormalized action reads

$$\mathcal{S}_\mathrm{r}(\pi) = \frac{Z Z_f}{2} \int \mathrm{d}^d x \left[(\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2 \right] - \mu^2 f \int \sigma(x) \mathrm{d}^d x\,, \tag{44.43}$$

in which σ is now given by:

$$\sigma(x) = \left[f^2 Z^{-1} - \pi^2 \right]^{1/2}.$$

Thus in terms of the renormalized fields Δ_2^{-1} gets an extra factor $Z_f Z$, $V^{(4)}$ a factor $Z_f Z^2$ and $V^{(6)}$ a factor $Z_f Z^3$. Moreover μ^2 is replaced by $\mu^2 Z^{-1/2} Z_f^{-1}$.

14.2.3. *One-loop order calculations.* Calculate the vacuum expectation value $\langle \sigma \rangle$ and determine the field renormalization constant. Calculate the inverse two-point function $\Gamma^{(2)}_{ij} = \left[W^{(2)} \right]^{-1}_{ij}$ and determine the coupling constant renormalization.

Solution: *The vacuum expectation.* $\langle \sigma \rangle$ is given by

$$= f Z^{-1/2} - \tfrac{1}{2} f^{-1} \left\langle \pi^2 \right\rangle = f \left(Z^{-1/2} - \frac{3}{2f^2} I \right),$$

where I is an integral which appears everywhere at one-loop

$$I=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d p}{p^2+\mu^2}=\mu^{d-2}\frac{1}{(4\pi)^{d/2}}\Gamma(1-d/2)\,.$$

If we then choose

$$Z=1+\frac{3g}{2\pi\varepsilon}+O\left(g^2\right),$$

we get $\langle\sigma\rangle=f$.

The two-point function. The two-point function is proportional to δ_{ij} which we omit in what follows. Then

$$\Gamma^{(2)}(p)=ZZ_f p^2+Z^{1/2}\mu^2+\frac{1}{f^2}\left(p^2+\tfrac{3}{2}\mu^2\right)\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q}{q^2+\mu^2}+O\left(f^{-4}\right).$$

We verify that the value of Z obtained above renders the coefficient of μ^2 finite. From the coefficient of p^2 we find that if we choose

$$Z_f=1-\frac{2g}{2\pi\varepsilon}\,,$$

the two-point is finite and reduces to its tree approximation

$$\Gamma^{(2)}(p)=p^2+\mu^2+O\left(f^{-4}\right).$$

14.2.4. Calculate the connected amputated four-point function $W^{(4)}_{\text{amp.}}$ at one-loop. In what follows we adopt the convention that all momenta enter so that $p_1+p_2+p_3+p_4=0$. Verify that the solution is finite in two dimensions.

Solution: For symmetry reasons the four-point function $\tilde W^{(4)}_{\text{amp.}}$ has the form

$$\left(\tilde W^{(4)}_{\text{amp.}}\right)_{ijkl}=C\left(p_1,p_2,p_3,p_4\right)\delta_{ij}\delta_{kl}+C\left(p_1,p_3,p_2,p_4\right)\delta_{ik}\delta_{jl}+C\left(p_1,p_4,p_3,p_2\right)\delta_{il}\delta_{kj}\,, \tag{44.44}$$

where $C\left(p_1,p_2,p_3,p_4\right)$ is symmetric in the exchange $p_1\leftrightarrow p_2$ with $p_3\leftrightarrow p_4$ and $\{p_1,p_2\}\leftrightarrow\{p_3,p_4\}$. In what follows we only give C.

It is convenient to introduce the notation

$$p_{ij}=p_i+p_j\,.$$

In the tree approximation $\Gamma^{(4)}$ is immediately obtained from the four-point vertex:

$$C=-\frac{1}{f^2}\left(Z_f Z^2 p_{12}^2+Z^{3/2}\mu^2\right)=-\frac{1}{f^2}\left[\left(1+\frac{4g}{2\pi\varepsilon}\right)p_{12}^2+\left(1+\frac{9}{2}\frac{g}{2\pi\varepsilon}\right)\mu^2\right].$$

At one-loop both the four-point and the six-point interactions contribute. For what concerns the six-point vertex we have just to contract two external legs. One finds

$$\delta_1 C=-\frac{1}{f^4}\left(5p_{12}^2+p_1^2+p_2^2+p_3^2+p_4^2+\tfrac{13}{2}\mu^2\right)I.$$

Then we have the contributions of two four-point vertices. The following identity is useful

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q\, q_\mu q_\nu}{\left[(q+p/2)^2+\mu^2\right]\left[(q-p/2)^2+\mu^2\right]} = S\delta_{\mu\nu} + T p_\mu p_\nu\,,$$

with

$$\begin{aligned}(d-1)S &= \tfrac{1}{2}I - \tfrac{1}{4}\left(p^2+4\mu^2\right)B(p)\\ T &= \frac{1}{(d-1)p^2}\left[\tfrac{1}{2}(d-2)I + \tfrac{1}{4}\left(p^2+4\mu^2\right)B(p)\right]\end{aligned}$$

where we defined the bubble diagram B

$$B(p) = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(q^2+\mu^2)\left[(p+q)^2+\mu^2\right]}.$$

Note the useful relation

$$\tfrac{1}{2}(d-2)I = -\mu^2 B(0).$$

We then find that C gets four contributions, which in the picture of faithful diagrams correspond to a bubble, a triangle and a square box direct and crossed. The sum of the two first contributions is

$$f^4\delta_2 C = \tfrac{3}{2}\left(p_{12}^2+\mu^2\right)^2 B\left(p_{12}\right) + \left(p_{12}^2+\mu^2\right)\left[2I - \left(p_{12}^2 - \tfrac{1}{2}\left(p_1^2+p_2^2+p_3^2+p_4^2\right)\right)B\left(p_{12}\right)\right].$$

The direct box yields

$$\begin{aligned}f^4\delta_3 C = \frac{1}{d-1}&\Bigg[\left(p_{14}^2-p_1^2-p_4^2\right) - \tfrac{1}{2}(d-2)\Bigg(p_{13}^2+p_1^2+p_4^2-p_2^2-p_3^2\\
&-\frac{(p_1^2-p_3^2)(p_2^2-p_4^2)}{p_{13}^2}\Bigg)\Bigg] I + \left(p_1^2+p_4^2\right)I - \frac{1}{4(d-1)}\left(p_{13}^2+4\mu^2\right)\\
&\times\left[2\left(p_{14}^2-p_1^2-p_4^2\right) + \left(p_{13}^2+p_1^2+p_4^2-p_2^2-p_3^2 - \frac{(p_1^2-p_3^2)(p_2^2-p_4^2)}{p_{13}^2}\right)\right]\\
&\times B\left(p_{13}\right) + \tfrac{1}{4}\left(p_{13}^2-p_1^2-p_3^2\right)\left(p_{13}^2-p_2^2-p_4^2\right)B\left(p_{13}\right).\end{aligned}$$

The terms have been written the way they appear in the calculation. They can be slightly rearranged

$$\begin{aligned}f^4\delta_3 C = \frac{1}{d-1}&\left[p_{14}^2 + \tfrac{1}{2}(d-2)\left(p_1^2+p_2^2+p_3^2+p_4^2-p_{13}^2+\frac{(p_1^2-p_3^2)(p_2^2-p_4^2)}{p_{13}^2}\right)\right] I\\
&-\frac{\left(p_{13}^2+4\mu^2\right)}{4(d-1)}\left(2p_{14}^2+p_{13}^2-p_1^2-p_2^2-p_3^2-p_4^2-\frac{(p_1^2-p_3^2)(p_2^2-p_4^2)}{p_{13}^2}\right)\\
&\times B\left(p_{13}\right) + \tfrac{1}{4}\left(p_{13}^2-p_1^2-p_3^2\right)\left(p_{13}^2-p_2^2-p_4^2\right)B\left(p_{13}\right).\end{aligned}$$

The last contribution is simply $\delta_3 C(p_1,p_2,p_4,p_3)$.

After renormalization the total one-loop contribution can be written

$$\begin{aligned}f^4\delta C = \tfrac{1}{2}&\left(p_{12}^2+\mu^2\right)\left(p_{12}^2+p_1^2+p_2^2+p_3^2+p_4^2+3\mu^2\right)B\left(p_{12}\right)\\
&+\frac{1}{4(d-1)}\left(2p_{14}^2+p_{13}^2-p_1^2-p_2^2-p_3^2-p_4^2-\frac{(p_1^2-p_3^2)(p_2^2-p_4^2)}{p_{13}^2}\right)\\
&\times\left[4\mu^2 B(0) - \left(p_{13}^2+4\mu^2\right)B\left(p_{13}\right)\right] + \tfrac{1}{4}\left(p_{13}^2-p_1^2-p_3^2\right)\left(p_{13}^2-p_2^2-p_4^2\right)B\left(p_{13}\right)\\
&+\frac{1}{4(d-1)}\left(2p_{13}^2+p_{14}^2-p_1^2-p_2^2-p_3^2-p_4^2-\frac{(p_1^2-p_4^2)(p_2^2-p_3^2)}{p_{14}^2}\right)\\
&\times\left[4\mu^2 B(0) - \left(p_{14}^2+4\mu^2\right)B\left(p_{14}\right)\right] + \tfrac{1}{4}\left(p_{14}^2-p_1^2-p_4^2\right)\left(p_{14}^2-p_2^2-p_3^2\right)B\left(p_{14}\right),\end{aligned}$$

expression finite for all $d < 4$.

14.2.5. Write and verify the consequences of WT identities which involve only the one to four-point 1PI correlation functions, using equation (14.34).

Solution. It is convenient to introduce some additional notation to take into account the $O(3)$ symmetry. Let us set

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} G(p),$$
$$\tilde{\Gamma}^{(3)}_{ij}(p_1, p_2; p_3) = \delta_{ij} V(p_1, p_2; p_3),$$

in which our conventions are that indices correspond of course to π-fields, and in mixed π–σ correlation functions the arguments of the π-fields are placed first.

In the non-linear σ-model we have the relation

$$\tilde{\Gamma}^{(4)}_{ijkl}(p_1, p_2, p_3, p_4) = \left(\tilde{W}^{(4)}_{\text{amp.}}\right)_{ijkl}(p_1, p_2, p_3, p_4) ,$$

for which we use the parametrization (44.44).

They are two ways to derive WT identities for correlation functions. One can either start from the WT identities for connected correlation functions which are the same as for the linear σ-model, or from the WT identities for the 1PI functional. In the latter case differentiating respectively with respect first to π (and setting $H(x) = c$), and then π and H one finds, after Fourier transformation,

$$G(0)\langle\sigma\rangle = c, \tag{44.45}$$
$$\tilde{\Gamma}^{(2)}(p)\tilde{\Gamma}^{(2)}_{ij}(p) + \delta_{ij} = \langle\sigma\rangle\,\tilde{\Gamma}^{(3)}_{ij}(0, -p; p). \tag{44.46}$$

The second equation can be rewritten

$$\tilde{\Gamma}^{(2)}(p)G(p) + 1 = \langle\sigma\rangle\,V(0, -p; p). \tag{44.47}$$

Finally differentiating three times with respect to π we find a relation between 3 and four-point correlation functions:

$$-\langle\sigma\rangle\,\tilde{\Gamma}^{(4)}_{ijkl}(0, p_2, p_3, p_4) + \tilde{\Gamma}_{ij}(p_2)\tilde{\Gamma}^{(3)}_{kl}(p_3, p_4; p_2) + \text{ 2 terms} = 0\,, \tag{44.48}$$

which implies

$$\langle\sigma\rangle\,C(0, p_2, p_3, p_4) = G(p_2)\,V(p_3, p_4; p_2)\,.$$

Setting $p_2^2 = -m_\pi^2$ one recovers from the last equation Adler's consistency conditions

$$C\left(0, p_2(p_2^2 = -m_\pi^2), p_3, p_4\right) = 0\,.$$

Eliminating V between eqs. (44.47) and (44.48) one derives Weinberg's relation

$$\frac{\partial}{\partial p^2}\left((\langle\sigma\rangle)^2 C(0, p, 0, -p) + G(p)\right)\Big|_{p^2=-m_\pi^2} = 0\,.$$

The verification of these identities on the explicit expressions is the straightforward.

A few additional details can be found in

D. Bessis and J. Zinn-Justin, *Phys. Rev.* D5 (1972) 1313; J. Zinn-Justin, in the proceedings of Les Houches summer school 1971, C. Itzykson ed., Gordon and Breach, New-York 1973,

where the model is studied as a limit of the linear σ-model.

44.15 General Non-linear Models in Two Dimensions : Chapter 15

Exercise 15.1

Show by explicit calculation that the differential operators

$$\mathcal{D}_\alpha \equiv D_a^\alpha \frac{\partial}{\partial \pi_a} = \left[t_{ab}^\alpha \pi_b + t_{aj}^\alpha \sigma_j(\pi)\right] \frac{\partial}{\partial \pi_a}\,, \tag{44.49}$$

where the functions σ_i are defined in equation (15.19), satisfy the commutation relations (15.42).

Solution: Let us first verify that equation (15.42) is indeed equivalent to equation (15.43):

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = D_a^\alpha \frac{\partial}{\partial \pi_a} D_b^\beta \frac{\partial}{\partial \pi_b} - D_a^\beta \frac{\partial}{\partial \pi_a} D_b^\alpha \frac{\partial}{\partial \pi_b}.$$

The only surviving terms correspond to the most left derivatives acting on D and thus

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = D_a^\alpha \frac{\partial D_b^\beta}{\partial \pi_a} \frac{\partial}{\partial \pi_b} - D_a^\beta \frac{\partial D_b^\alpha}{\partial \pi_a} \frac{\partial}{\partial \pi_b}.$$

With the explicit form (44.49) we find

$$D_a^\alpha \frac{\partial D_b^\beta}{\partial \pi_a} = \left[t_{ac}^\alpha \pi_c + t_{aj}^\alpha \sigma_j\right] \left(t_{ba}^\beta + t_{bk}^\beta \frac{\partial \sigma_k}{\partial \pi_a}\right).$$

Antisymmetrizing in α, β we find commutators of generators t^α except for the terms proportional to $\partial\sigma_k/\partial\pi_a$. However we then use equation (15.29). This reduce again the calculation to commutation relations and we find the anticipated result.

Exercise 15.2

Verify then by direct calculation the covariance of equations (15.43) in a reparametrization of the homogeneous space.

Solution. Let χ^i be another set of coordinates on the manifold. Then the infinitesimal group transformations take the form

$$\delta_\omega \chi^i = \frac{\partial \chi^i}{\partial \phi^j} D_\alpha^j \omega_\alpha \equiv \partial_j \chi^i D_\alpha^j \omega_\alpha\,.$$

We want thus to prove that the new functions Δ_α^i:

$$\Delta_\alpha^i = \partial_j \chi^i D_\alpha^j\,,$$

satisfy the commutation relations (15.43).

$$\begin{aligned}\Delta_\alpha^j \frac{\partial \Delta_\beta^i}{\partial \chi^j} &= \partial_k \chi^j D_\alpha^k \left[\partial_l \chi^j\right]^{-1} \left(\partial_l \partial_m \chi^i D_\beta^m + \partial_m \chi^i \partial_l D_\beta^m\right) \\ &= D_\alpha^l \left(\partial_l \partial_m \chi^i D_\beta^m + \partial_m \chi^i \partial_l D^m \beta\right).\end{aligned}$$

We now see that the first term is symmetric in the exchange $\alpha \leftrightarrow \beta$ and thus cancels in the commutator, while the second term is proportional to the term in (15.43). The result follows. Actually using the results of Chapter 22, one can verify that, as in the case of the curl of a vector, the relation (15.43) can be rewritten by replacing the derivative by a covariant derivative when the Christoffel connection is symmetric.

Exercise 15.3

Prove that the tensor g^{ij} $(A15.14)$ satisfies the relation $(A15.7)$ and is therefore a symmetric inverse metric tensor.

Solution. Let us verify the identity $(A15.7)$ for the tensor

$$g^{ij} = D^i_\beta(\varphi)D^j_\beta(\varphi).$$

In terms of g^{ij} it reads

$$\begin{aligned} 0 &= D^i_\alpha \frac{\partial g^{jk}}{\partial \varphi^i} - g^{jl}\frac{\partial D^k_\alpha}{\partial \varphi^l} - g^{kl}\frac{\partial D^j_\alpha}{\partial \varphi^l} \\ &= D^k_\beta\left(D^i_\alpha \partial_i D^j_\beta - D^i_\beta \partial_i D^j_\alpha\right) + D^j_\beta\left(D^i_\alpha \partial_i D^k_\beta - D^i_\beta \partial_i D^k_\alpha\right). \end{aligned}$$

We now use the commutation relations (15.43) we find

$$0 = D^k_\beta f_{\alpha\beta\gamma} D^j_\gamma + D^j_\beta f_{\alpha\beta\gamma} D^k_\gamma\,,$$

which is satisfied because the structure constants are antisymmetric in $\beta \leftrightarrow \gamma$.

Exercise 15.4

Verify explicitly in the case of the $O(4)$ group that the tensor $(A15.14)$ is the inverse of the metric on the sphere S_3.

Solution. The Lie algebra of $O(4)$ is the same as the Lie algebra of $O(3) \times O(3)$. The infinitesimal transformations of parameters ω, ω' can be written

$$\delta\pi_i = \epsilon_{ijk}\omega_j\pi_k + \omega'_i\sigma\,, \quad \Rightarrow \quad \delta\sigma = -\omega'_i\pi_i\,,$$

where we have chosen π_i and $\sigma = \sqrt{1-\pi^2}$ as the coordinates on the sphere. We thus find

$$\gamma^{ij} = \sum_k \left(\epsilon_{ikl}\pi_l\epsilon_{jkm}\pi_m + \delta_{ik}\sigma\delta_{jk}\sigma\right) = \delta_{ij} - \pi_i\pi_j\,,$$

or calculating the inverse

$$g_{ij} = \delta_{ij} + \pi_i\pi_j/(1-\pi^2),$$

in which we recognize the metric on S_3.

44.16 Abelian Gauge Theories : Chapter 18

Exercise 16.1

In the notation of Section 18.4.2 verify explicitly the compatibility of Gauss's law with quantum evolution in the case of a non-vanishing external current.

Solution: Let us differentiate Gauss's law

$$\frac{1}{i}\partial_i \frac{\delta}{\delta A_i(x)}\Psi(\mathbf{A}) = J_d(x,t)\Psi(\mathbf{A})\,,$$

with respect to time, and use Schrödinger's equation

$$\frac{1}{i}\partial_i \frac{\delta}{\delta A_i(x)}\mathcal{H}\Psi(\mathbf{A}) = i\dot{J}_d(x,t)\Psi(\mathbf{A}) + J_d(x,t)\mathcal{H}\Psi(\mathbf{A}). \tag{44.50}$$

We then use the property that in the absence of source the generator of time-independent gauge transformations commutes with the hamiltonian $\mathcal{H}$

$$\left[\frac{1}{i}\partial_i \frac{\delta}{\delta A_i(x)}, \mathcal{H}\right] = i\partial_i J_i \,.$$

After commutation in the l.h.s. of (44.50) and again application of Gauss's law we find:

$$-i\partial_i J_i \Psi(\mathbf{A}) = i\dot{J}_d(x,t)\Psi(\mathbf{A}),$$

which follows from current conservation.

Exercise 16.2

For the one-loop calculations which follow dimensional regularization will be used. For the trace of the unit matrix in the γ matrix space one can take the 4D value: $\operatorname{tr}\mathbf{1} = 4$.

Establish the Feynman rules. Calculate the fermion two-point function at one-loop order, in the Feynman gauge, using the action (18.26). Infer the fermion mass and fermion field RG functions.

Solution: The photon propagator reads

$$\Delta_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2 + m^2}. \tag{44.51}$$

The fermion propagator reads:

$$\Delta_{\mathrm{F}}(p) = \frac{(M - i\not{p})}{p^2 + M^2}\,, \tag{44.52}$$

and finally there is the $\bar{\psi}\psi A$ vertex V_μ:

$$V_\mu\,(q; p_1, p_2) = -ie\gamma_\mu \,. \tag{44.53}$$

At one-loop order, in the Feynman gauge, and for a massless vector field, the fermion two-point function is then given by:

$$\Gamma^{(2)}(p) = M + i\not{p} + \frac{e^2}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{k^2}\gamma_\mu \left[M + i(\not{p} + \not{k})\right]^{-1}\gamma_\mu \,.$$

We then use the identities:

$$\gamma_\mu\gamma_\mu = d\,, \quad \gamma_\mu \not{v}\gamma_\mu = (2-d)\not{v}\,.$$

Projecting then $\Gamma^{(2)}$ onto its components

$$\Gamma^{(2)}(p) = A\left(p^2\right) + i\not{p}B\left(p^2\right), \tag{44.54}$$

we can replace $\not{k}$ by $\not{p}k\cdot p/p^2$. We obtain:

$$A = M\left[1 + \frac{e^2 d}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{k^2\left(M^2 + (k+p)^2\right)}\right],$$

$$B = 1 + \frac{e^2(d-2)}{(2\pi)^d p^2}\int \frac{\mathrm{d}^d k\left(p^2 + k\cdot p\right)}{k^2\left(M^2 + (k+p)^2\right)}.$$

Replacing $M + i\not{p}$ by $Z_\psi(Z_M M + i\not{p})$ we derive the renormalization constants in the minimal subtraction scheme:

$$Z_M = 1 - 3e^2\frac{N_d}{\varepsilon}\,, \quad Z_\psi = 1 - e^2\frac{N_d}{\varepsilon}.$$

In the notation of Section 18.9 it follows ($\alpha = e^2/4\pi$):

$$\eta_M = \frac{3}{2\pi}\alpha\,, \quad \eta_\psi = \frac{1}{2\pi}\alpha\,.$$

Exercise 16.3

We now consider the QED action (18.26), with a mass term for the photon and quantized in a general covariant gauge

$$\mathcal{S}\left(A_\mu,\bar\psi,\psi\right)=\int \mathrm{d}^d x\left[\tfrac{1}{4}F_{\mu\nu}^2+\tfrac{1}{2}m^2A_\mu^2+\frac{1}{2\xi}\left(\partial_\mu A_\mu\right)^2-\bar\psi\left(\not{D}+M\right)\psi\right]. \qquad (44.55)$$

16.3.1. Write explicitly the WT identities involving the A two-point function and the $\bar\psi\psi A$ three-point function both for the 1PI and connected correlation functions and verify them in the tree approximation.

Solution: From identity (18.77) we derive

$$\begin{aligned}
k_\mu\Gamma^{(2)}_{\mu\nu}(k)&=k_\nu\left(k^2/\xi+m^2\right),\\
k_\mu\Gamma^{(3)}_\mu(k;p;q)&=e\Gamma^{(2)}(p+k;q)-e\Gamma^{(2)}(p;q+k),\\
k_\mu W^{(2)}_{\mu\nu}(k)&=\xi\frac{k_\nu}{k^2+\xi m^2},\\
\left(k^2/\xi+m^2\right)k_\mu W^{(3)}_\mu(k;p;q)&=eW^{(2)}(p+k;q)-eW^{(2)}(p;q+k).
\end{aligned}$$

The first equation implies

$$\Gamma^{(2)}_{\mu\nu}(k)=\Gamma_{\mathrm{T}}(k)\left(k^2\delta_{\mu\nu}-k_\mu k_\nu\right)+m^2\delta_{\mu\nu}+k_\mu k_\nu/\xi\,. \qquad (44.56)$$

This identity for $\Gamma^{(3)}_\mu$ can made even more explicit by parametrizing $\Gamma^{(2)}$ as:

$$\Gamma^{(2)}(p)=A\left(p^2\right)+i\not{p}B\left(p^2\right), \qquad (44.57)$$

where:

$$A=M+O(\alpha)\,,\quad B=1+O(\alpha).$$

16.3.2. Calculate the inverse fermion two-point function $\Gamma^{(2)}(p)$ at one-loop order, i.e. reduce it to a linear combination of scalar "bubble" integrals of the form

$$B\left(p;m_1,m_2\right)=\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\left(k^2+m_1^2\right)\left[(k-p)^2+m_2^2\right]}.$$

It is convenient to also introduce the integral $I(\mu)$

$$I(\mu)=\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2+\mu^2}\quad\Rightarrow\quad B(0;m_1,m_2)=\frac{1}{m_1^2-m_2^2}\left[I(m_2)-I(m_1)\right].$$

Solution: The inverse fermion two-point function $\Gamma^{(2)}(p)$ at one-loop order is

$$\Gamma^{(2)}(p)=M+i\not{p}+e^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\gamma_\mu\left(M+i\not{k}-i\not{p}\right)\gamma_\nu}{(k-p)^2+M^2}$$
$$\left(\frac{\delta_{\mu\nu}}{k^2+m^2}+\frac{(\xi-1)k_\mu k_\nu}{\left(k^2+m^2\right)\left(k^2+\xi m^2\right)}\right). \qquad (44.58)$$

We set

$$\Gamma^{(2)}(p) = MA_1(p) + i\not{p}A_2(p). \tag{44.59}$$

We then use a few standard tricks:

$$\gamma_\mu\gamma_\mu = d\,, \quad \gamma_\mu\not{p}\gamma_\mu = (2-d)\not{p}\,, \quad \not{k}\not{p}\not{k} = 2pk\not{k} - k^2\not{p}\,,$$
$$\int \mathrm{d}^d k\, \not{k} f\left(k^2, p^2, kp\right) = \not{p}\int \mathrm{d}^d k \frac{kp}{p^2} f\left(k^2, p^2, kp\right),$$

Repeated use of the identity

$$2pk = k^2 + m_1^2 - \left[(k-p)^2 + m_2^2\right] + p^2 + m_2^2 - m_1^2,$$

leads to

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2kp}{(k^2+m_1^2)\left[(k-p)^2+m_2^2\right]} = I\left(m_2\right) - I\left(m_1\right) + \left(p^2 + m_2^2 - m_1^2\right) B\left(p; m_1, m_2\right),$$
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(2kp)^2}{(k^2+m_1^2)\left[(k-p)^2+m_2^2\right]} = \left(3p^2 + m_2^2 - m_1^2\right) I\left(m_2\right)$$
$$- \left(p^2 + m_2^2 - m_1^2\right) I\left(m_1\right) + \left(p^2 + m_2^2 - m_1^2\right)^2 B\left(p; m_1, m_2\right).$$

It is easy to obtain the coefficient of M in eq. (44.59). Setting $m' = m\sqrt{\xi}$ one finds

$$A_1 = 1 + e^2\left[(d-1)B(p; M, m) + \xi B(p; M, m')\right].$$

Let us then write A_2 as the sum of terms

$$A_2 = 1 + \frac{e^2}{2p^2}\left(A_{21} + A_{22}\right),$$

where A_{21} corresponds to $\xi = 1$, i.e. is the result in the Feynman gauge, and A_{22} the remaining part. Using the previous remarks A_{21} can easily be evaluated

$$A_{21} = (2-d)\left[I(M) - I(m) - \left(p^2 - M^2 + m^2\right) B(p; M, m)\right].$$

The remaining contribution can be written

$$A_{22} = (\xi - 1)\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2k^2p^2 + 2(kp)k^2 - 4(kp)^2}{\left[(k-p)^2 + M^2\right]\left(k^2+m^2\right)\left(k^2+\xi m^2\right)}.$$

We then use the identities

$$\frac{(\xi-1)k^2}{\left(k^2+m^2\right)\left(k^2+\xi m^2\right)} = \frac{\xi}{k^2+\xi m^2} - \frac{1}{k^2+m^2}$$
$$\frac{(\xi-1)}{\left(k^2+m^2\right)\left(k^2+\xi m^2\right)} = \frac{1}{m^2}\left(\frac{1}{k^2+m^2} - \frac{1}{k^2+\xi m^2}\right).$$

We obtain

$$\begin{aligned}A_{22} = {} & \left\{\xi\left[I(M) - I(m') + \left(3p^2 + M^2 - m'^2\right) B(p; M, m')\right]\right.\\ & \left. - \left[I(M) - I(m) + \left(3p^2 + M^2 - m^2\right) B(p; M, m)\right]\right\} + (1-\xi)I(M)\\ & + \frac{1}{m^2}\left\{\left[-\left(p^2 + M^2 - m'^2\right) I(m') + \left(p^2 + M^2 - m'^2\right)^2 B(p; M, m')\right]\right.\\ & \left. + \left[\left(p^2 + M^2 - m^2\right) I(m) - \left(p^2 + M^2 - m^2\right)^2 B(p; M, m)\right]\right\}.\end{aligned}$$

The final expression for A_2 then is:

$$\begin{aligned}A_2 &= 1 + e^2 \frac{2-d}{2p^2} \left[I(M) - I(m) - \left(p^2 - M^2 + m^2\right) B(p; M, m)\right] \\ &+ \frac{e^2}{2m^2 p^2} \Big\{ \left(p^2 + M^2\right) [I(m) - I(m')] + \Big[\left(p^2 + M^2\right)^2 + \left(p^2 - M^2\right) m'^2\Big] \\ &\times B(p; M, m') - \Big[\left(p^2 + M^2\right)^2 + \left(p^2 - M^2\right) m^2\Big] B(p; M, m) \Big\} .\end{aligned}$$

16.3.3. Calculate in the bare theory the one-loop correction to the fermion mass. What can be said about the gauge dependence of the result. Explain.

Solution: At one-loop order the fermion mass M_{F} is given by

$$M_{\mathrm{F}} = M \frac{A_1(iM)}{A_2(iM)} .$$

A short calculation leads to

$$\frac{M_{\mathrm{F}}}{M} = 1 + e^2 \left\{ \frac{2-d}{2M^2} [I(M) - I(m)] + \left[2 + (d-2) \frac{m^2}{2M^2}\right] B(iM; M, m) \right\} .$$

One verifies that the gauge dependence has cancelled. Indeed the fermion mass is a physical observable and it has been shown that physical observables are gauge independent.

16.3.4. Characterize the divergences and calculate the corresponding renormalization constants. It is convenient to set $d = 4 - \varepsilon$ and

$$N_d = \frac{2}{\Gamma(d/2)(4\pi)^{d/2}} .$$

What can be said of the gauge dependence of Z_M and Z_ψ? Explain.

Solution: To introduce counterterms we have to replace $\not p$ by $Z_\psi \not p$ and M by $Z_\psi Z_M M$ in the tree approximation. The one-loop divergences of A_1 are given by:

$$[A_1]_{\text{div.}} = e^2 \frac{N_d}{\varepsilon} (3 + \xi),$$

and thus

$$Z_\psi Z_M = 1 - e^2 \frac{N_d}{\varepsilon} (3 + \xi).$$

For A_2 one finds

$$[A_2]_{\text{div.}} = e^2 \frac{N_d}{\varepsilon} \xi .$$

Thus

$$Z_\psi = 1 - e^2 \frac{N_d}{\varepsilon} \xi , \qquad Z_M = 1 - 3e^2 \frac{N_d}{\varepsilon} .$$

The fermion field is charged and thus the fermion correlation functions are gauge dependent. Its renormalization is therefore gauge independent. Moreover the gauge dependence is consistent with the analysis of Section 18.7. Instead the mass is coupled to the operator $\bar\psi(x)\psi(x)$ which is neutral and thus gauge invariant.

16.3.5. Write the regularized $A_\mu\bar{\psi}\psi$ vertex (1PI) at one-loop order and check the WT identity relating it to the fermion two-point function.

Solution: At one-loop order $\Gamma_\mu^{(3)}$ reads

$$\begin{aligned}\Gamma_\mu^{(3)}(q;p_1;p_2) &= -ie\gamma_\mu \\ &+ (-ie)^3\int\frac{\mathrm{d}^d k}{(2\pi)^d}\gamma_\nu\left[M+i(\not{k}+\not{p}_1)\right]^{-1}\gamma_\mu\left[M+i(\not{k}-\not{p}_2)\right]^{-1}\gamma_\rho[\Delta_\xi]_{\nu\rho}(k).\end{aligned}$$

Let us now calculate the one-loop contribution to $q_\mu\Gamma_\mu^{(3)}(q;p_1;p_2)$. We note

$$q+p_1+p_2=0 \ \Rightarrow\ i\not{q}=\left[M+i(\not{k}-\not{p}_2)\right]-\left[M+i(\not{k}+\not{p}_1)\right].$$

It follows:

$$\begin{aligned}q_\mu\Gamma_\mu^{(3)}(q;p_1;p_2) &= e\left[(M+i\not{p}_1)-(M-i\not{p}_2)\right] \\ &+ e^3\int\frac{\mathrm{d}^d k}{(2\pi)^d}\gamma_\nu\left\{\left[M+i(\not{k}+\not{p}_1)\right]^{-1}-\left[M+i(\not{k}-\not{p}_2)\right]^{-1}\right\}\gamma_\rho[\Delta_\xi]_{\nu\rho}(k).\end{aligned}$$

We recognize in the r.h.s. the difference between two inverse propagators.

Calculation of the vertex. Let us set

$$\Gamma_\mu^{(3)}(q;p_1;p_2) = -ie\gamma_\mu + (-ie)^3 C_\mu(q;p_1;p_2).$$

If we specialize to the Feynman gauge ($\xi=1$) C_μ becomes

$$C_\mu=\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\gamma_\nu\big(M-i(\not{k}+\not{p}_1)\big)\gamma_\mu\big(M-i(\not{k}-\not{p}_2)\big)\gamma_\nu}{\left[(k+p_1)^2+M^2\right](k^2+m^2)\left[(k-p_2)^2+M^2\right]}.$$

We then write C_μ as the sum of three terms:

$$C_\mu = M^2C_{1\mu}+iMC_{2\mu}+C_{3\mu}\,,$$

where the denominators of the integrands of C_1, C_2, C_3 successively are

$$\begin{aligned}&(2-d)\gamma_\mu\,,\\ &(2-d)\left[(\not{k}+\not{p}_1)\gamma_\mu+\gamma_\mu(\not{k}-\not{p}_2)\right]-2\gamma_\mu(\not{k}+\not{p}_1)-2(\not{k}-\not{p}_2)\gamma_\mu\,,\\ &-2\gamma_\mu(\not{k}-\not{p}_2)(\not{k}+\not{p}_1)+2(\not{k}+\not{p}_1)(\not{k}-\not{p}_2)\gamma_\mu-(2-d)(\not{k}+\not{p}_1)\gamma_\mu(\not{k}-\not{p}_2).\end{aligned}$$

44.17 Gauge Theories: Master Equation and Renormalization : Chapter 21

These exercises try to illustrate the methods which have been used in the quantization of non-abelian gauge theories. However, one should be warned that the abelian example has some simplifying features which are not shared by the general non-abelian case.

We want to study the quantization and renormalization of the free abelian gauge field action,

$$\mathcal{S}\left(A_\mu\right)=\tfrac{1}{4}\int \mathrm{d}^d x\, F_{\mu\nu}^2(x), \tag{44.60}$$

where

$$F_{\mu\nu}(x)=\partial_\mu A_\nu-\partial_\nu A_\mu\,,$$

in the covariant gauge

$$G(A)\equiv\partial_\mu A_\mu+ieA_\mu^2=0\,. \tag{44.61}$$

Exercise 17.1

17.1.1. Write the quantized action. Note that after quantization the action is no longer free.

Solution: Because the gauge variation of the gauge condition (44.61) is field-dependent

$$\delta A_\mu=\partial_\mu\Lambda\ \Rightarrow\ \delta G(A)=\partial^2\Lambda+2ieA_\mu\partial_\mu\Lambda\,,$$

the Faddeev–Popov determinant is no longer a constant and it is necessary to introduce ghost fields. To express the constraint it is also necessary to introduce a Lagrange multiplier field $\lambda(x)$. The quantized action can then be written:

$$\mathcal{S}(A,\lambda,\bar C,C)=\int \mathrm{d}^d x\left[\tfrac{1}{4}F_{\mu\nu}^2+\lambda\left(\partial_\mu A_\mu+ieA_\mu^2\right)-C\left(\partial^2+2ieA_\mu\partial_\mu\right)\bar C\right]. \tag{44.62}$$

Note the relation

$$\partial^2+2ieA_\mu\partial_\mu-\left(\partial_\mu+ieA_\mu\right)^2=-ie\left(\partial_\mu A_\mu+ieA_\mu^2\right)=0\,.$$

17.1.2. Verify the BRS symmetry of the quantized action.

Solution: It is straightforward to verify that the action (44.62) is invariant under the BRS transformations

$$\delta A_\mu=\varepsilon\partial_\mu\bar C\,,\quad \delta\bar C=0\,,\quad \delta C=\varepsilon\lambda\,,\quad \delta\lambda=0\,.$$

17.1.3. Derive the corresponding WT identities.

Solution: One immediately notes a drastic simplification with respect to the case of non-abelian gauge symmetries: The BRS transformations are linear in the fields and therefore no sources for composite fields are required. If we add to the action (44.62) the following source terms

$$\mathcal{S}_{\text{sources}}=-\int \mathrm{d}^d x\left(J_\mu A_\mu+K\lambda+\bar C\eta+\bar\eta C\right),$$

we find that the generating functional $\mathcal{W}$ for connected correlation functions satisfies

$$\int \mathrm{d}^d x\left(J_\mu(x)\partial_\mu\frac{\delta}{\delta\eta(x)}+\bar\eta(x)\frac{\delta}{\delta K(x)}\right)\mathcal{W}(J,K,\bar\eta,\eta)=0\,.$$

Because the BRS transformations are linear the Legendre transformation is straightforward. The 1PI functional Γ satisfies

$$\mathcal{D}\Gamma(A_\mu,\lambda,\bar{C},C)=0 \quad \text{with} \quad \mathcal{D}\equiv\int \mathrm{d}^d x\left(\partial_\mu \bar{C}\frac{\delta}{\delta A_\mu(x)}+\lambda(x)\frac{\delta}{\delta C(x)}\right). \qquad (44.63)$$

17.1.4. Verify that the additional terms in the action generated by the quantization procedure are themselves the BRS variation of a local quantity, i.e. that they can be written $\mathcal{D}\Phi(A,\lambda,\bar{C},C)$.

Solution: The additional terms in (44.62) generated by the quantization procedure are the BRS variation of Φ:

$$\begin{aligned}\Phi(A,\lambda,\bar{C},C)&=\int \mathrm{d}^d x\, C(x)G(A)\\ \Rightarrow\ \mathcal{D}\Phi&=\int \mathrm{d}^d x\left[\lambda\left(\partial_\mu A_\mu+ieA_\mu^2\right)-C\left(\partial^2+2ieA_\mu\partial_\mu\right)\bar{C}\right]. \end{aligned} \qquad (44.64)$$

Exercise 17.2

17.2.1. Calculate the various propagators.

Solution: The $C\bar{C}$ propagator is $1/k^2$ in momentum space. The fields A_μ and λ are coupled in the gaussian approximation and the propagators thus form a 2×2 matrix:

$$\begin{pmatrix}\left(\delta_{\mu\nu}-\dfrac{k_\mu k_\nu}{k^2}\right)\dfrac{1}{k^2} & i\dfrac{k_\mu}{k^2}\\ -i\dfrac{k_\mu}{k^2} & 0\end{pmatrix} \qquad (44.65)$$

17.2.2. Derive from the WT identities the general form of the renormalized action.

Solution: First let us note that the large momentum behaviour of the propagators shows that the quantized action (44.62) is renormalizable by power counting in four dimensions and that the canonical dimensions of fields are

$$d=4\ \Rightarrow\ [A]=1\,,\quad [\lambda]=2\,,\quad [\bar{C}]+[C]=2\,.$$

Let us then write the most general form of the renormalized action just taking into account power counting and ghost number conservation. A term quartic in the ghost fields is allowed but by power counting it is necessarily proportional to $[C(x)\bar{C}(x)]^2$ and thus vanishes because ghosts are fermions. Then

$$\mathcal{S}_\mathrm{r}=\Sigma(A)+\int \mathrm{d}^d x\left[-\tfrac{1}{2}a\lambda^2+\lambda G_\mathrm{r}(A,\bar{C},C)-CL(A)\bar{C}\right], \qquad (44.66)$$

where G_r has dimension two and is thus at most linear in $C\bar{C}$, $L(A)$ has also dimension two and a is a constant.

We have in full generality shown that if the 1PI functional Γ satisfies linear WT identities the renormalized action satisfies the same. Thus

$$\int \mathrm{d}^d x\left(\partial_\mu \bar{C}\frac{\delta}{\delta A_\mu(x)}+\lambda(x)\frac{\delta}{\delta C(x)}\right)\mathcal{S}_\mathrm{r}(A_\mu,\lambda,\bar{C},C)=0\,.$$

We now introduce the decomposition (44.66) and identify the coefficients of powers of λ:

$$\lambda^2 : \ \frac{\delta G_{\mathrm{r}}}{\delta C} = 0\,, \qquad \lambda : \ \frac{\delta G_{\mathrm{r}}}{\delta A_\mu}\partial_\mu \bar{C} - L(A)\bar{C} = 0\,,$$
$$\int \mathrm{d}^d x\, C\, \partial_\mu \bar{C} \frac{\delta L(A)}{\delta A_\mu(x)}\bar{C} = 0\,, \qquad \int \mathrm{d}^d x\, \partial_\mu \bar{C} \frac{\delta \Sigma(A)}{\delta A_\mu(x)} = 0\,.$$

From the first equation it follows that G_{r} is only a function of A_μ and from the second that $L(A)$ is determined as a function of G_{r}. The most general form of G_{r} compatible with power counting is

$$G_{\mathrm{r}} = Z_\lambda \left(\partial_\mu A_\mu + ieZ_e A_\mu^2 + Z_G\right).$$

Thus the renormalized ghost action is

$$\mathcal{S}_{\mathrm{ghosts}} = -\int \mathrm{d}^d x\, Z_\lambda C(x) \left(\partial^2 + 2ieZ_e A_\mu \partial_\mu\right) \bar{C}(x).$$

The third equation now is automatically satisfied and the last equation implies that $\Sigma(A)$ is gauge invariant. The renormalized action thus reads (with the change in notation $a \mapsto \xi_0 Z_\lambda^2$)

$$\begin{aligned}\mathcal{S}_{\mathrm{r}} = \int \mathrm{d}^d x\, \big[& Z_A \tfrac{1}{4} F_{\mu\nu}^2 - \tfrac{1}{2}\xi_0 Z_\lambda^2 \lambda^2 + Z_\lambda \lambda \left(\partial_\mu A_\mu + ieZ_e A_\mu^2 + Z_G\right) \\ & - Z_\lambda C \left(\partial^2 + 2ieZ_e A_\mu \partial_\mu\right)\bar{C}\big]\,. \end{aligned} \tag{44.67}$$

17.2.3. What can be said about the stability under renormalization of the gauge condition $G(A) = 0$? Integrate with the renormalized action over the λ-field.

Solution: We see in action (44.67) that a term proportional to λ^2 has been generated by renormalization. The condition $G(A)$ is thus not stable under renormalization but is replaced by a gaussian distribution centred around $G(A) = 0$. Since the action is quadratic in λ the gaussian integral over λ results in an action where λ is replaced by the solution of the corresponding field equation:

$$Z_\lambda \lambda = \left(\partial_\mu A_\mu + ieZ_e A_\mu^2 + Z_G\right)/\xi_0\,,$$

and thus

$$\mathcal{S}_{\mathrm{r}}(A, \bar{C}, C) = \int \mathrm{d}^d x \left[Z_A \tfrac{1}{4} F_{\mu\nu}^2 + \tfrac{1}{2}\left(\partial_\mu A_\mu + ieZ_e A_\mu^2\right)^2/\xi_0 - Z_\lambda C \left(\partial^2 + 2ieZ_e A_\mu \partial_\mu\right)\bar{C}\right]. \tag{44.68}$$

17.2.4. Study the variation of the gauge field two-point function $W^{(2)}_{\mu\nu}$ under an infinitesimal change δe of the bare parameter e in the action (44.62). It is suggested to use the result (44.64), i.e. express the variation of $W^{(2)}_{\mu\nu}$ in terms of the variation of Φ. An integration by parts will then lead to the result.

Solution: The induced variation of Φ is

$$\delta\Phi = i\delta e \int \mathrm{d}^d x\, C(x) A_\mu^2(x).$$

The variation of the generating functional $\mathcal{Z}(J)$ of gauge field correlation functions follows:

$$\delta\mathcal{Z}(J) = -i\delta e \int [\mathrm{d}A\mathrm{d}\lambda\mathrm{d}\bar{C}\mathrm{d}C] \exp\left[-\mathcal{S}(A,\lambda,\bar{C},C) + \int \mathrm{d}^d y\, J_\mu A_\mu\right] \mathcal{D} \int \mathrm{d}^d x\, A_\mu^2(x) C(x).$$

We then integrate by parts to let $\mathcal{D}$ act on the action and source:

$$\begin{aligned}\delta\mathcal{Z}(J) &= i\delta e \int [\mathrm{d}A\mathrm{d}\lambda\mathrm{d}\bar{C}\mathrm{d}C] \int \mathrm{d}^d x\, A_\mu^2(x) C(x) \mathcal{D} \exp\left[-\mathcal{S}(A,\lambda,\bar{C},C) + \int \mathrm{d}^d y\, J_\mu A_\mu\right] \\ &= i\delta e \int [\mathrm{d}A\mathrm{d}\lambda\mathrm{d}\bar{C}\mathrm{d}C] \int \mathrm{d}^d x\, A_\mu^2(x) C(x) \int \mathrm{d}^d y\, \partial_\mu \bar{C}(y) J_\mu(y) \\ &\quad \times \exp\left[-\mathcal{S}(A,\lambda,\bar{C},C) + \int J_\mu A_\mu\right].\end{aligned}$$

Differentiating then twice with respect to J and setting $J=0$ we obtain the variation of the two-point function:

$$\delta W_{\mu\nu}^{(2)}(x,y) = i\delta e \left\langle \int \mathrm{d}^d z\, A_\rho^2(z) C(z) A_\mu(x) \partial_\nu^y \bar{C}(y) \right\rangle,$$

where the average symbol means functional average with the weight $\mathrm{e}^{-\mathcal{S}}$. Covariance and translation invariance tell us that the two-point function $\left\langle \int \mathrm{d}^d z\, A_\rho^2(z) C(z) A_\mu(x) \bar{C}(y) \right\rangle$ must be of the form

$$\left\langle \int \mathrm{d}^d z\, A_\rho^2(z) C(z) A_\mu(x) \bar{C}(y) \right\rangle = \partial_\mu f(x-y).$$

It follows after Fourier transformation that the variation of $W_{\mu\nu}^{(2)}$ has only a longitudinal component and thus the transverse part is independent of e. Therefore

$$\tilde{W}_{\mu\nu}^{(2)}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2} + k_\mu k_\nu w(k,e).$$

In particular $Z_A = 1$.

44.18 Classical and Quantum Gravity : Chapter 22

Exercise 18.1

To the reader who is puzzled by the Wheeler–DeWitt equation we propose to study the quantization of the relativistic point-particle of space-time coordinates x^μ (the metric being Minkowskian, i.e. $+ - - \cdots$) in an external electromagnetic field A_μ. We consider the following lagrangian

$$-\mathcal{L}(x^\mu, \dot{x}^\mu, N) = \tfrac{1}{2} N^{-1}(\tau) \dot{x}^\mu \dot{x}_\mu + \tfrac{1}{2} N(\tau) m^2 + e A_\mu(x) \dot{x}^\mu, \qquad (44.69)$$

where τ parametrizes the particle trajectory and $N(\tau)$ is a metric factor which ensures the independence of the theory on the trajectory parametrization. Vectors have components $x^0, x^1, \ldots$.

This theory has a reparametrization invariance:

$$\tau \mapsto \tau' , \qquad N'(\tau') = N(\tau)\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} .$$

Of course it is possible to eliminate N from the lagrangian by solving the corresponding equation of motion

$$m^2 N^2 = \dot{x}^\mu \dot{x}_\mu .$$

One obtains

$$-\mathcal{L}'\left(x^\mu, \dot{x}^\mu\right) = \pm m \left(\dot{x}^\mu \dot{x}_\mu\right)^{1/2} + eA_\mu(x)\dot{x}^\mu. \tag{44.70}$$

This lagrangian has still a reparametrization invariance.

Discuss quantization from both points of view ($N(\tau) \equiv 1$ defines the temporal gauge).

Solution: By a change in the parametrization we can set N to 1. It is then possible to construct a hamiltonian

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -\dot{x}_\mu - eA_\mu(x),$$

and thus

$$\mathcal{H}(p,x) = p_\mu \dot{x}^\mu - \mathcal{L} = \tfrac{1}{2}\left[m^2 - (p+eA)^2\right] .$$

In addition we must impose the N equation of motion as a constraint:

$$\dot{x}^\mu \dot{x}_\mu = m^2 \ \Rightarrow\ \mathcal{H} = 0 .$$

After quantization we thus look for wave functions $\psi(x)$ which satisfy

$$\left[(p+eA)^2 - m^2\right]\psi = 0 .$$

We recognize the Klein–Gordon equation which is, in this framework, the analogue of the Wheeler–DeWitt equation.

In this example we recognize that τ is not really a time but simply an auxiliary parameter, and the absence of evolution in this parameter is not too surprising: the Klein–Gordon equation is the real dynamical equation.

If we consider instead the lagrangian $\mathcal{L}'$ we could have the illusion that conjugated momenta p_μ can be defined because $\mathcal{L}'$ depends on all time derivatives:

$$p_\mu = \frac{\partial \mathcal{L}'}{\partial \dot{x}^\mu} = \mp m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} - eA_\mu .$$

However we note that this equation implies

$$(p_\mu + eA_\mu)^2 = m^2 ,$$

and therefore the conjugated momenta are not independent. We thus instead define a temporal gauge by the condition

$$x^0(\tau) \equiv \tau .$$

We then restrict ourselves again to space components and introduce the notation $\hat{p}, \hat{A}$ for the vectors of components $p_\mu, A_\mu, \mu > 0$, and $\hat{v}$ for $\dot{x}^\mu, \mu > 0$. We find

$$\hat{p} = \pm \frac{m\hat{v}}{\sqrt{1-\hat{v}^2}} - e\hat{A} \ \Rightarrow\ \pm m\hat{v} = \frac{\hat{p}+e\hat{A}}{\sqrt{(\hat{p}+e\hat{A})^2 + m^2}} .$$

The hamiltonian follows

$$\mathcal{H} = \pm\sqrt{(\hat{p}+e\hat{A})^2 + m^2} + eA_0 .$$

We see that if we take into account the two signs we recover the Klein–Gordon equation. We note also that only one sign corresponds to a hamiltonian bounded from below.

Exercise 18.2

The exercises which follow try to motivate the description of discretized 2D quantum gravity in terms of triangulated surfaces as explained in the appendix A22.

18.2.1. *Random walks.* One considers brownian random walks on a hypercubic lattice with constant decay rate per unit time $1-a$ or survival probability a, $0<a<1$. Calculate the probability $U(y,x)$ for a walk starting from point x to pass through the point y at any later time. Study the scaling limit in which $|x-y|$ becomes large and a approaches 1.

Solution. Let us call $P_n(z)$ the probability to be at point z at time n. It satisfies a recursion relation (in d dimensions)

$$P_n(z)=\frac{a}{2d}\sum_{\mu=1}^{2d}P_{n-1}(z-e_\mu),$$

where the points $z-e_\mu$ are all the neighbours of z on the lattice. The Fourier transform $\tilde{P}_n(k)$ of $P_n(z)$ then satisfies

$$\tilde{P}_n(k)=\frac{a}{d}\left(\sum_{\mu=1}^{d}\cos k_\mu\right)\tilde{P}_{n-1}(k)\qquad P_0(z)=\delta(z-x).$$

After a short calculation we find

$$U(y,x)=(1-a)\sum_{n=0}P_n(y)=\frac{1}{(2\pi)^d}(1-a)\int \mathrm{d}^d k\frac{\mathrm{e}^{ik(y-x)}}{1-(a/d)\sum_\mu \cos k_\mu}.$$

In the long distance scaling limit the integral is dominated by the small values of k. The probability is non-vanishing only if a is close to 1. We thus set $a=1-m^2/2$ and expand the cosines

$$1-(a/d)\sum_\mu \cos k_\mu\sim\tfrac{1}{2}(m^2+k^2).$$

It follows that $U(y,x)$ has the form of a propagator of a scalar particle of mass m:

$$U(y,x)=\frac{1}{(2\pi)^d}m^2\int \mathrm{d}^d k\frac{\mathrm{e}^{ik(y-x)}}{k^2+m^2}.\tag{44.71}$$

This interpretation of the euclidean propagator associates random walks to Feynman diagrams.

18.2.2. One can write directly in the continuum a path integral which weights the paths according to their length as in the preceding discretized example. One can integrate with the action

$$\mathcal{S}(\mathbf{x})=m\int \mathrm{d}t\,\sqrt{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}.\tag{44.72}$$

We recognize a form analogous to expression (44.70). With this action, however, one again faces the problem of reparametrization invariance. Therefore study first the continuum limit of a natural discretization of this expression, the probability distribution $U(y,x)$ being given by

$$U(y,x)=\delta(x-y)+\sum_{n=1}^{\infty}\int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\ldots\mathrm{d}^d x_n\exp\left[-\mu\sum_{i=0}^{n}|x_i-x_{i+1}|\right],$$

with $x_0 = x$, $x_{n+1} = y$, and μ being a constant.

Solution. The expression can be rewritten

$$U(y,x) = \sum_n \langle y|T^n|x\rangle = \langle y|[1-T]^{-1}|x\rangle\,,$$

with

$$\langle y|T|x\rangle = \mathrm{e}^{-\mu|y-x|}\,.$$

After a Fourier transformation the operator acts by multiplication:

$$\left\langle p'|\tilde{T}|p\right\rangle = \delta^{(d)}(p'-p)\tilde{T}(p),$$

with

$$\tilde{T}(p) = \int \mathrm{d}^d x\ \mathrm{e}^{ipx}\,\mathrm{e}^{-\mu|x|}\,.$$

The method of calculation is quite similar to the preceding example. The continuum limit again is obtained when the parameter μ is adjusted in such a way that the lowest eigenvalue $\tilde{T}(0)$ approaches 1. The long distance asymptotic scaling form depends only on the expansion of $\tilde{T}(p)$ for p small:

$$\tilde{T}(p) = 1 - K(m^2+p^2) + O\left(p^4\right).$$

We recover the Feynman propagator (44.71) and note that the result is quite robust (using the language of critical phenomena we can call it "universal") since it does not depend on the explicit form of $\langle y|T|x\rangle$ except that it is a function of $|x-y|$ and decreases fast enough at large separation.

18.2.3. Recover now the preceding result from the continuum action (44.72).

Solution. We face the problem of parametrization and moreover at first sight we know of no direct recipe to calculate a path integral with such an action.

To solve the problem of the reparametrization invariance we can choose the length along the path s:

$$(\mathrm{d}s)^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,.$$

The path integral then can be formally written

$$U(y,x) = \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-ms}\int_{x(0)=x}^{x(s)=y}[\mathrm{d}x(\sigma)]\delta\left(\dot{\mathbf{x}}(\sigma)\cdot\dot{\mathbf{x}}(\sigma)-1\right).$$

As usual we introduce a Fourier representation for the δ-function

$$U(y,x) = \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-ms}\int[\mathrm{d}x(\sigma)][\mathrm{d}\lambda(\sigma)]\exp\left[-\tfrac{1}{2}\int_0^s \mathrm{d}\sigma\left(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}-1\right)\lambda(\sigma)\right], \qquad (44.73)$$

where the integration over λ runs parallel to the imaginary axis. We now change variables $\sigma \mapsto t$, setting

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \lambda(\sigma).$$

The action in (44.73) becomes

$$\int_0^s \mathrm{d}\sigma\,(\dot{\mathbf{x}}(\sigma)\cdot\dot{\mathbf{x}}(\sigma)-1)\,\lambda(\sigma)=\int_0^T \mathrm{d}t\,\left[\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2-\lambda^2\right],$$

with $0=t(\sigma=0)$, $T=t(s)$. Using then

$$s=\int_0^s \mathrm{d}\sigma=\int_0^T \lambda\bigl(\sigma(t)\bigr)\mathrm{d}t\,,$$

we find

$$\begin{aligned}U(y,x)=&\int_0^\infty \mathrm{d}T\int_{x(0)=x}^{x(T)=y}[\mathrm{d}x(t)][\mathrm{d}\lambda(t)]\lambda(T)\\&\times\exp\left[-\int_0^T \mathrm{d}t\,\left(\tfrac{1}{2}\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t)-\tfrac{1}{2}\lambda^2(t)+m\lambda(t)\right)\right].\end{aligned}$$

The integral over λ is gaussian and can be performed. We finally obtain

$$U(y,x)=m\int_0^\infty \mathrm{d}T\int_{x(0)=x}^{x(T)=y}[\mathrm{d}x(t)]\exp\left[-\tfrac{1}{2}\int_0^T \mathrm{d}t\,\left(\dot{\mathbf{x}}(t)\cdot\dot{\mathbf{x}}(t)+m^2\right)\right].$$

Under this form we recognize a natural representation of the propagator (44.71) in the continuum, when we consider the operator $-\mathrm{d}^2+m^2$ as a quantum hamiltonian:

$$U(y,x)=m\int_0^\infty \mathrm{d}T\,\langle y|\,\mathrm{e}^{-TH}\,|x\rangle=m\,\langle y|\,H^{-1}\,|x\rangle\,,\quad H=-\mathrm{d}^2+m^2.$$

However the geometrical meaning in terms of a length of a path and the invariance under parametrization are no longer obvious. Returning to the first exercise we understand that this action results from the invariant action integral of the lagrangian (44.69) (for $A=0$) by gauge fixing.

44.19 Mean Field Theory : Chapter 24

Exercise 19.1

The spin 1 Ising model. Consider the Ising model with a n.n. ferromagnetic two-body interaction in which the spin S can take three values $\{0,1,-1\}$, the common weight of the configurations $\{1,-1\}$ being p and the weight of the last configuration being therefore $1-2p$. Study the model in the mean field approximation as a function of temperature and p.

Solution. We use everywhere the notation of Chapter 24. We find

$$A(\phi)=-\ln(\lambda+\cosh\varphi)\,,\qquad \text{with } \lambda=1/2p-1>0\,.$$

The magnetization then is

$$M=-A'(\varphi)=\frac{\sinh\varphi}{\lambda+\cosh\varphi}\,.$$

Hence $B(M)$, the Legendre transform of $-A(\varphi)$ satisfies

$$B'(M)=\varphi(M)=\ln\left(\frac{\lambda M+\sqrt{1-M^2+\lambda^2M^2}}{1-M}\right).$$

From this expression we derive

$$B''(M)=\frac{1}{1-M^2}\left(1+\frac{\lambda}{\sqrt{1-M^2+\lambda^2M^2}}\right),$$
$$B(M)=\tfrac{1}{2}(1+M)\ln D(M)+\tfrac{1}{2}(1-M)\ln D(-M)\,,$$

with

$$D(M)=\frac{\left(\lambda^2-1\right)M-1+\lambda\sqrt{1-M^2+\lambda^2M^2}}{\lambda^2-1}\,.$$

The thermodynamical potential $\Gamma(M)$ is given by

$$\Gamma(M)=-VM^2+B(M).$$

Note that the cases $p=\frac{1}{2}$ ($\lambda=0$) and $p=\frac{1}{4}$ ($\lambda=1$) are identical in the mean field approximation, they correspond to $l=1$ and $l=2$ respectively in the notation of Section 24.3.1.

It is useful to expand $B(M)$ up to order M^6:

$$B(M)=(1+\lambda)\left[\tfrac{1}{2}M^2+\tfrac{1}{24}(1+\lambda)(2-\lambda)M^4+\tfrac{1}{240}(1+\lambda)^2\left(8-9\lambda+3\lambda^2\right)M^6\right]+O\left(M^8\right).$$

Finally we note

$$B''(M)\geq B''(0)\quad\text{for }\lambda\leq 2\,.$$

We immediately conclude that for $\lambda<2$ or $p>1/6$ $\Gamma(M)$ remains convex until V reaches the critical value V_c

$$V_c=\tfrac{1}{2}(1+\lambda)=1/(4p),$$

which defines the critical temperature of a second order phase transition. The line of second order phase transition terminates at the special point $\lambda=2$ which corresponds to different physics and is called a tricritical point. For $V-V_c$ and M small we can expand $\Gamma(M)$

$$\Gamma(M)=(V_c-V)M^2+\tfrac{9}{40}M^6+O\left(M^8\right).$$

The tricritical point is characterized by new exponents like $\beta=1/4$, $\delta=5$, new universal ratios like

$$C_+/C_-=4\,.$$

Finally the whole scaling region is not limited to $\lambda=\lambda_c=2$ but contains also $|\lambda-\lambda_c|$ small. After rescaling of temperature, field, magnetization and the additional thermodynamical variable $\lambda-\lambda_c$ the scaling equation of state can be cast into the universal form

$$H=\frac{\partial\Gamma}{\partial M}=M^5f(x,y),\quad f(x,y)=1+x+y$$
$$x=(V-V_c)/M^4,\qquad y=(\lambda_c-\lambda)/M^2\,.$$

Exercise 19.2

The antiferromagnetic Ising model. We consider an Ising-like system on a hypercubic lattice with antiferromagnetic interactions:

$$-\beta\mathcal{H} = -\sum_{ij} V_{ij} S_i S_j \,, \quad V_{ij} \geq 0\,,\; V_{ii} = 0\,.$$

The lattice can be decomposed into two sublattices corresponding to odd or even sites (as defined by the parity of the sum of components of the position vector). We characterize the sublattices by a sign $\epsilon_i = (-1)^i$ assigned to each site i. We then assume

$$V_{ij}\epsilon_i\epsilon_j = -V_{ij}\,,$$

a condition satisfied by the nearest neighbour interaction.

Define an order parameter for the antiferromagnetic transition. Characterize then the effect of a small constant magnetic field within the mean field approximation.

Solution. In the absence of a magnetic field, the antiferromagnetic Ising model on a bipartite lattice (a lattice which can be decomposed into two sublattices such that in the two-spin interaction the two spins belong to different sublattices) can be simply related to a ferromagnetic Ising model by setting

$$\sigma_i = \epsilon_i S_i\,. \tag{44.74}$$

In the new variables the interaction reads

$$-\beta\mathcal{H} = \sum_{ij} V_{ij}\sigma_i\sigma_j\,,$$

and is thus ferromagnetic. The alternating spins σ_i define an order parameter for the antiferromagnetic system.

In mean field theory the thermodynamic potential $\Gamma(M_i)$ reads

$$\Gamma(M_i) = -\sum_{ij} V_{ij} M_i M_j + \sum_i B(M_i).$$

A constant magnetic field after the transformation (44.74) becomes an alternating field $\epsilon_i H$. In such a field the minimum of the potential corresponds to the sum a constant and alternating magnetization

$$M_i = M + \epsilon_i \widetilde{M}\,.$$

The equation

$$H\epsilon_i = \frac{\partial\Gamma}{\partial M_i}$$

for small field H reduces to two equations

$$\begin{aligned} H &= 2V\widetilde{M} + a\widetilde{M} + O\left(\widetilde{M}^3, \widetilde{M}M^2\right) \\ 0 &= -2VM + aM + \tfrac{1}{6}bM^3 + \tfrac{1}{2}b\widetilde{M}^2 M + O\left(M^5, \widetilde{M}M^3, \widetilde{M}^4 M\right), \end{aligned}$$

We see in the first equation that $\widetilde{M}$ always remains linear in H

$$\widetilde{M} \sim H/(2V+a).$$

Substituting this solution in the second equation we observe that the only effect, at leading order, is to modify the critical temperature

$$V_c - \tfrac{1}{2}a \sim \tfrac{1}{16}bH^2/a^2.$$

The analysis confirms that since a constant magnetic field is not coupled to the order parameter, it does not destroy the phase transition, but simply modifies non-universal quantities, like the critical temperature.

Exercise 19.3

Mean field theory and effective Landau–Ginzburg action for models with Z_N symmetry. One now considers a lattice model on a cubic lattice with short range ferromagnetic interaction in which the two-body interaction has a Z_N symmetry. For example one can represent the degrees of freedom at each site i by a phase θ_i which takes N discrete values:

$$\theta = 2\pi n/N, \quad p = 0, \ldots, N-1\,.$$

The configuration energy then takes the form:

$$-\beta\mathcal{H} = \sum_{i,j} V_{ij}\cos\left(\theta_i - \theta_j\right),$$

with:

$$V_{ii} = 0\,, \quad V_{ij} \geq 0\,.$$

The N configurations on a site have of course, due to the symmetry, the same weight.

Note that for $N = 2$ one simply recovers the Ising model. For $N = 3$ one finds a model which is equivalent to the 3-state Potts model. Define an order parameter for the Z_N model, $N > 1$. Discuss the model in the mean field approximation, in particular the order of the phase transition.

Solution. An order parameter for the system is the expectation value $M = \left\langle e^{i\theta}\right\rangle$. The variable M is thus a complex number, on which the symmetry group acts by multiplication. Symmetry arguments tell us that the thermodynamical potential, is a real function of the invariant polynomials M^*M, which is $U(1) \equiv O(2)$ invariant, M^N and M^{*N}. If we assume the existence of a second order phase transition, we can expand the potential for M small. Several cases then have to be considered:

(i) If $N \geq 5$ then the terms which are not functions only of M^*M can be neglected and the scaling theory is the same as for the $O(2)$ symmetric model (a model with a continuous symmetry). Only corrections to the leading terms reveal the difference. Note however that the correction terms play here a more important role than usual, since they suppress below T_c the would-be Goldstone modes of the $O(2)$ symmetry.

(ii) If $N = 4$ instead we find the most general model depending on two order parameters M_1, M_2 and symmetric in $M_1 \leftrightarrow M_2$, which is equivalent to the $O(2)$ model with cubic anisotropy.

(iii) For $N = 3$ a cubic term is allowed. Then we expect generically a first order phase transition.

Let us now calculate the expansion of the function $A(\varphi, \varphi^*)$ up the fourth order:

$$e^{-A(\varphi,\varphi^*)} = \frac{1}{N}\sum_n \exp\left(e^{2in\pi/N}\,\varphi + e^{-2in\pi/N}\,\varphi^*\right),$$

and thus

$$-A(\varphi,\varphi^*) = \varphi^*\varphi + \tfrac{1}{6}\left(\varphi^3 + \varphi^{*3}\right)\delta_{N3} + \tfrac{1}{24}\left(\varphi^4 + \varphi^{*4}\right)\delta_{N4} - \tfrac{1}{4}\left(\varphi^*\varphi\right)^2.$$

After Legendre transformation

$$-A(\varphi,\varphi^*) + B(M, M^*) = M^*\varphi + M\varphi^*\,,$$

we obtain

$$B(M,M^*) = M^*M + \tfrac{1}{4}\left(M^*M\right)^2 - \tfrac{1}{6}\left(M^3 + M^{*3}\right)\delta_{N3} - \tfrac{1}{24}\left(M^4 + M^{*4}\right)\delta_{N4}\,.$$

The thermodynamical potential at fourth order then is

$$\Gamma(MM^*) = VM^*M - B(M,M^*),$$

confirming the general analysis.

Other results. Th $N = 3$ model is known to have a second order phase transition in 2 dimensions (with exponents derived from CFT). In 3 dimensions the nature of the transition (first or second order) has been an object of debate and of heavy numerical investigations. The motivation was the following: The Z_N group is the centre of the $SU(N)$ group and is involved in the question of confinement (Wilson's loop arguments). The confinement phase transition at high physical temperature in lattice QCD in four dimensionsthus is related to the phase transition of the Z_3 model in 3 dimensions (the physical gauge group is $SU(3)$). Most recent data suggests that the transition is first order.

Exercise 19.4

The q-state Potts model. The model is defined in terms of spin variables which can take q values with equal weight and a configuration energy of the form:

$$-\beta\mathcal{H} = \sum_{\text{n.n.}} J\delta_{S_i S_j}\,.$$

For $q = 2$ the Ising model is recovered. Study the model for higher values of the integer q. A general review on the q-state Potts model and the percolation problem ($q = 1$) can be found in

F.Y. Wu, *Rev. Mod. Phys.* 54 (1982) 235.

44.20 General Renormalization Group : Chapter 25

Exercise 20.1

Study with the formalism of Chapter 25 the tricritical ϕ^6 field theory near three dimensions. We recall that the tricritical point is defined by:

$$\Gamma^{(2)}(p=0) = 0\,,\quad \Gamma^{(4)}(p_i=0) = 0\,.$$

Calculate in particular the RG functions at leading order. We write the interaction $g\Lambda^{6-2d}\phi^6/6!$ and set $\varepsilon = 6 - 2d$. It is advised to use dimensional regularization.

Extend this analysis to the $u\phi^{2N}/(2N)!$ field theory near the dimension in which it is renormalizable (the upper-critical dimension). It is suggested to set $\varepsilon = 2N - d(N-1)$.

Solution. The weight factor for the $2N$-point function at leading ($N-1$-loop) order is $(2N-1)!/(N!)^2/(N-1)!$. For the two-point function one finds $1/(2N-1)!$. These factors correspond to the symmetry factors of the corresponding diagrams. To then calculate the diagrams in the massless theory it is convenient to use real space variables because the diagrams are just powers of the propagator. One starts from

$$\frac{1}{p^2} = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\int \mathrm{d}^d x\, x^{2-d}\,\mathrm{e}^{-ipx}\,.$$

Calling $K(d)$ the combination $\Gamma(d/2-1)\pi^{-d/2}/4$ one finds for the diagram with $m+1$ propagator attached to two points in which flows a momentum p:

$$D_{m+1}(p) = K^{m+1}(d)\pi^{d/2}\frac{\Gamma\bigl(1-m(d-2)/2\bigr)}{\Gamma\bigl((m+1)(d-2)/2\bigr)}\left(\frac{2}{p}\right)^{2-m(d-2)}.$$

The $2N$-point function corresponds to $m=N-1$ and the two-point function to $m=2N-2$.

For $N=3$ $K(3)=1/(4\pi)$. The weight of the contribution to the six-point function is $5/3$ and the divergent part then $5C/3\varepsilon$ with $C=1/(16\pi^2)$. For the two-point function the results are respectively $1/120$ and $C^2/(12\varepsilon)$. After a short calculation one obtains the RG functions at leading order

$$\beta(g) = -\varepsilon g + C\frac{5}{3}g^2, \qquad \eta(g) = C^2\frac{1}{720}g^2.$$

Exercise 20.2

Long range forces. From a mean field analysis one can show that O(N) symmetric models with two-body long range forces yield, near a continuous phase transition, a $(\boldsymbol{\phi}^2)^2$ field theory with however an unusual, non-local, propagator. For a potential V decaying at large distance r as $V(r)\propto r^{-d-\sigma}$ the effective hamiltonian reads

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}^d x\left[\tfrac{1}{2}\boldsymbol{\phi}\cdot(-\Delta)^{\sigma/2}\boldsymbol{\phi} + \tfrac{1}{2}r\boldsymbol{\phi}^2 + \frac{g}{4!}\left(\boldsymbol{\phi}^2\right)^2\right],$$

with $\sigma<2$. This means that in momentum space the propagator is

$$\Delta_{ij}(k) = \frac{\delta_{ij}}{k^\sigma}.$$

20.2.1. Find the upper-critical dimension, the equivalent of dimension four for the short range case.

Solution. The canonical dimension of the field ϕ is now $\frac{1}{2}(d-\sigma)$. The dimension of the coupling constant u is then $d-4(d-\sigma)/2=2\sigma-d$. The upper-critical dimension is then $d=2\sigma$.

20.2.2. What can be said about the field strength renormalization and the exponent η, and the limit $\sigma\to 2$?

Solution. The UV divergences of perturbation theory generate counterterms which are polynomials in the fields and their derivatives. Therefore no term proportional to $\phi\cdot(-\Delta)^{\sigma/2}\phi$ can be generated: The field strength is not renormalized, $Z_\phi=1$, and the critical two-point function remains proportional to $k^{-\sigma}$. Note that, as a consequence, the limit $\sigma\to 2$ cannot coincide with the result at $\sigma=2$.

20.2.3. Calculate the β-function at one-loop order and the exponent ν at leading order in the ε-expansion.

Solution. We set $u=\Lambda^{2\sigma-d}g$. The function $\beta(g)$ comes from the divergent part of a unique bubble diagram. One finds

$$\beta(g) = -\varepsilon g + \tfrac{N+8}{6}N_{2\sigma}g^2 + O\left(g^3,\varepsilon g^2\right), \qquad N_{2\sigma} = 2(4\pi)^{-\sigma}/\Gamma(\sigma).$$

At this order the calculation of Z_2 and thus η_2 involves the same diagram. One thus finds

$$\begin{aligned}\eta_2(g^*) &= -\varepsilon\tfrac{N+2}{N+8}\\ \nu^{-1} &= \sigma+\eta_2=\sigma-\varepsilon\tfrac{N+2}{N+8}\\ \gamma &= \nu\sigma = 1+\frac{\varepsilon}{\sigma}\frac{N+2}{N+8}\,.\end{aligned}$$

44.21 Scaling Laws in the Critical Domain : Chapter 26

Exercise 21.1

Discuss a general situation in which the RG β-function has the form of a gradient flow

$$\beta_i(g)=\frac{\partial V}{\partial g_i}\,. \tag{44.75}$$

Solution. First note that the gradient flow property is parametrization dependent. The properties we derive here will be valid more generally for all equations which are obtained from a gradient flow by reparametrization.

The corresponding flow of the coupling constant takes the form

$$\frac{\mathrm{d}g_i}{\mathrm{d}\ln\lambda}=\beta_i(g)=\frac{\partial V}{\partial g_i}\,. \tag{44.76}$$

Let us multiply the equation by $\mathrm{d}g_i/\mathrm{d}\ln\lambda$. Then

$$\left(\frac{\mathrm{d}g_i}{\mathrm{d}\ln\lambda}\right)^2=\frac{\partial V}{\partial g_i}\frac{\mathrm{d}g_i}{\mathrm{d}\ln\lambda}=\frac{\mathrm{d}V}{\mathrm{d}\ln\lambda}>0\,.$$

We recover the essential property of a gradient flow: fixed points are minima or maxima of the potential V; pathologies like limiting cycles are excluded. Moreover if we calculate the matrix B of derivatives at a fixed point we find

$$B_{ij}=\frac{\partial\beta_i}{\partial g_j}=\frac{\partial^2 V}{\partial g_i\partial g_j}\,,$$

which is a symmetric matrix. Therefore all its eigenvalues are real.

Exercise 21.2

Renormalization Group for models with cubic symmetry. Consider a model in which the order parameter is a N-component field $\boldsymbol{\phi}(x)$ which has a cubic symmetry:

$$\phi_i\leftrightarrow\phi_j\,,\quad \phi_i\leftrightarrow-\phi_i\,.$$

The Landau–Ginzburg action $\mathcal{S}(\boldsymbol{\phi})$ expanded up to fourth order in $\boldsymbol{\phi}$ then takes the form:

$$\mathcal{S}(\boldsymbol{\phi})=\int\mathrm{d}^d x\left[\tfrac{1}{2}\left(\partial_\mu\boldsymbol{\phi}\right)^2+\tfrac{1}{2}r\boldsymbol{\phi}^2+\frac{g}{4!}\left(\boldsymbol{\phi}^2\right)^2+\frac{\lambda}{4!}\sum_{i=1,N}\phi_i^4\right].$$

The term which breaks the $O(N)$ symmetry of the action may be generated in a ferromagnetic model by the interaction between the spin degrees of freedom with the lattice which has only cubic spatial symmetry.

21.2.1. Write the conditions relating g and λ which guaranty the stability of the classical potential.

Solution. The stability conditions are: $g+\lambda \geq 0$, $Ng+\lambda \geq 0$, the second relation relying on Schwartz's inequality.

21.2.2. Calculate the RG β-functions at leading order.

Solution. The two β functions are

$$\beta(g) = -\varepsilon g + \frac{1}{8\pi^2}\left(\frac{N+8}{6}g^2 + g\lambda\right)$$
$$\beta(\lambda) = -\varepsilon\lambda + \frac{1}{8\pi^2}\left(2g\lambda + \frac{3}{2}\lambda^2\right)$$

21.2.3. Find the various IR fixed points and discuss their stability.

Solution. One finds four fixed points, the gaussian fixed point $g=\lambda=0$, the Ising fixed point $g=0, \lambda = 16\varepsilon\pi^2/3$, the isotropic fixed point $\lambda=0, g=48\varepsilon\pi^2/(N+8)$ and a new cubic fixed point

$$g = \frac{16\pi^2\varepsilon}{N}, \qquad \lambda = \frac{16\pi^2(N-4)\varepsilon}{3N}.$$

Note that for $N \geq 1$ the cubic fixed point satisfies the stability condition $g+\lambda \geq 0$. Also all fixed points have $g \geq 0$. The lines $\lambda = 0$ and $g=0$ cannot be crossed.

To find the local stability of the four fixed points one has to calculate the eigenvalues of the matrix M of derivatives of the β-functions

$$M = -\varepsilon + \frac{1}{8\pi^2}\begin{pmatrix} \frac{N+8}{3}g+\lambda & g \\ 2\lambda & 2g+3\lambda \end{pmatrix}.$$

The set of eigenvalues are

$$\begin{array}{rl} \text{gaussian} & -\varepsilon\,, -\varepsilon\,, \\ \text{Ising} & -\frac{1}{3}\varepsilon\,, \varepsilon\,, \\ \text{isotropic} & \dfrac{4-N}{N+8}\varepsilon\,, \varepsilon, \\ \text{cubic} & \varepsilon, \dfrac{N-4}{3N}\varepsilon\,. \end{array}$$

We recover a particular example of a more general result, the isotropic fixed point is stable for $N < N_c$ with $N_c = 4 + O(\varepsilon)$. The cubic fixed point is stable only for $N > N_c$. At $N = N_c$ the two fixed points merge and exchange role. The Ising fixed point has always one direction of instability. Note that for some initial values of g, λ (like $g < 0$) the effective coupling constants are not attracted to a fixed point but run away. It is generally believed that this signals a first order phase transition.

21.2.4. Relate the eigenvalues of the stability matrix at the Ising model fixed point to Ising model exponents.

Solution. The perturbation of the Ising model by the isotropic $O(N)$ symmetric interaction involves the operators $(\boldsymbol{\phi}^2)^2$. At the fixed point all $\boldsymbol{\phi}$ components are decoupled. Therefore the $(\boldsymbol{\phi}^2)^2$ is equivalent to a linear combination of ϕ_1^4 and $\phi_1^2\phi_2^2$ which factorizes. The corresponding exponents are ω_{Ising} and $(\alpha/\nu)_{\text{Ising}}$.

44.22 Corrections to Scaling Behaviour : Chapter 27

The tricritical $(\boldsymbol{\phi}^2)^3$ field theory is also IR free in three dimensions. Verify that the RG equations for the $n\phi$, $l\phi^2$, $k\phi^4$ correlation functions can be written:

$$\left\{\Lambda\frac{\partial}{\partial\Lambda}+\beta(v)\frac{\partial}{\partial v}-k\eta_4(v)u\frac{\partial}{\partial u}-l\left[\eta_2(v)t+\eta_{24}(v)u^2\right]\frac{\partial}{\partial t}-\frac{n}{2}\eta(v)\right\}\Gamma^{(k,l,n)}=0\,,$$

except for $l=n=0, k\le 3$. Calculate the RG functions and study the logarithmic deviations from mean field theory. A review of tricritical behaviour can be found in D. Lawrie and S. Sarbach, in *Phase Transitions and Critical Phenomena* vol. 9, C. Domb and J.L. Lebowitz eds. (Academic Press, London 1984).

44.23 Non-Magnetic Systems and the $(\phi^2)^2$ Field Theory : Chapter 28

To the reader who finds the relation between the polymer model and the ϕ^4 field theory somewhat mysterious the following exercise is proposed: Calculate at one or two-loop order $Z^{(2)}(k,t)$ directly by expanding the measure (28.1) in powers of g. In addition to the trick which has already been used in the gaussian case the following remark is useful: Write a Fourier representation for the chain self-interaction term:

$$\delta^d\left(\mathbf{r}\left(u_1\right)-\mathbf{r}\left(u_2\right)\right)=\int\frac{\mathrm{d}^d q}{(2\pi)^d}\exp\left[i\mathbf{q}\cdot\int_0^s\mathrm{d}u\,(\theta(u_1-u)-\theta(u_2-u))\,\dot{\mathbf{r}}(u)\right]$$

where $\theta(u)$ is the Heaviside step function. To recover the Feynman diagrams of the ϕ^4 field theory it is suggested to integrate over $\mathbf{r}(u)$, then over all u variables and finally to Laplace transform, keeping the q-integrations which come from the Fourier representation for the end. Some additional inspiration can be found in Section 32.1.

Solution.

Polymers. Let us calculate the two-point function $Z^{(2)}(k,s)$ at one-loop order. At leading order we have found

$$\begin{aligned}
&G^{(2)}(k,s)=\langle\mathrm{e}^{i\mathbf{k}\cdot[\mathbf{r}(s)-\mathbf{r}(0)]}\rangle=\mathrm{e}^{-k^2|s|}\\
\Rightarrow\;\; &Z^{(2)}(k,s)=\int_0^\infty\mathrm{d}s\;\mathrm{e}^{-st}\,G^{(2)}(k,s)=\frac{1}{k^2+t}\,.
\end{aligned}$$

The one-loop contribution to $Z^{(2)}(k,t)$ reads

$$\begin{aligned}
\left[Z^{(2)}(k,t)\right]_{1\,\mathrm{loop}}&=-\frac{g}{6}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\int_0^\infty\mathrm{d}s\;\mathrm{e}^{-st}\,\zeta(k,q,s),\\
\zeta(k,q,s)&=\int_0^s\mathrm{d}u_1\mathrm{d}u_2\,\langle\mathrm{e}^{i\mathbf{k}\cdot[\mathbf{r}(s)-\mathbf{r}(0)]}\,\mathrm{e}^{i\mathbf{q}\cdot[\mathbf{r}(u_2)-\mathbf{r}(u_1)]}\rangle\\
&=2\int_0^s\mathrm{d}u_2\int_0^{u_2}\mathrm{d}u_1\,\mathrm{e}^{-k^2(s-u_2+u_1)-(k+q)^2(u_2-u_1)}\,.
\end{aligned}$$

We have used the property that the average of a product of exponentials is given by a simple gaussian integral, as we discuss in 32.1.

It is then convenient to change variables, $s, u_2, u_1 \mapsto s', u', u_1$:

$$s - u_2 + u_1 = s' , \qquad u_2 - u_1 = u' .$$

The integrals over u_1, s', u' can be explicitly performed. Shifting q in the remaining integral, $q + k \mapsto q$, we then obtain

$$Z^{(2)}(k,t) = \frac{1}{k^2+t} - \frac{g}{3}\frac{1}{(k^2+t)^2}\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{q^2+t} ,$$

which is indeed the one-loop contribution to the two-point function in the $(\phi^2)^2$ field theory in the $N = 0$ limit.

44.24 The $(\phi^2)^2$ Field Theory in the Large N Limit : Chapter 30

Exercise 24.1

Discuss the logarithmic corrections to mean field theory at the upper-critical dimension $d = 4$ in the large N expressions of Chapter 30. Consider more specifically the correlation length, the equation of state and the specific heat.

Solution. The correlation length above T_c is given by the limit of equation (30.23):

$$\tau \sim \frac{N}{8\pi^2} m^2 \ln(\Lambda/m) \ \Rightarrow\ \xi^2 = m^{-2} \sim \frac{N}{16\pi^2}\frac{\ln(\Lambda^2/\tau)}{\tau} .$$

One verifies that this result is in agreement with the analysis of Section 27.2. Similarly the thermodynamic potential density becomes

$$\mathfrak{G}(M) = \tfrac{1}{2} m^2 \left(\tau + M^2\right) - \frac{N}{32\pi^2} m^4 \ln(\Lambda/m).$$

m^2 is solution of the equation

$$2\frac{\partial \mathfrak{G}}{\partial m^2} = \tau + M^2 - \frac{N}{8\pi^2} m^2 \ln(\Lambda/m) = 0 .$$

The equation of state (equation (30.35)) thus becomes

$$H = m^2 M \sim \frac{16\pi^2}{N}\frac{M(M^2+\tau)}{\ln[\Lambda^2/(M^2+\tau)]} ,$$

again in agreement with the large N limit of the general result (27.20). Finally the specific heat above T_c is proportional to $1/\ln m$, i.e. at leading order to $1/\ln \tau$ as expected.

We stress again that even in four dimensions the bare coupling constant has played no role and the same results would have obtained from the $O(N)$ non-linear σ-model.

Exercise 24.2

Use the expressions the saddle point equations (30.11) for a potential $NU(\phi^2/N)$ and look for a tricritical point.

Solution. Let us explore further the equations (30.11) for a general potential U

$$m^2\sigma = 0\,, \tag{44.77a}$$

$$U'(\rho) - \tfrac{1}{2}m^2 = 0\,, \tag{44.77b}$$

$$\sigma^2/N - \rho + \Omega_d(m) = 0\,, \tag{44.77c}$$

($\Omega_d(m)$ is defined by equation (30.12).) At low temperature $\sigma \neq 0$ and thus m vanishes. The equation (44.77c) has a solution only for $\rho > \rho_c$,

$$\rho_c = \Omega_d(0), \;\Rightarrow\; \sigma = \sqrt{N}\sqrt{\rho - \rho_c}\,.$$

The equation (44.77b) then yields the critical temperature. Setting $U(\rho) = W(\rho) + \frac{1}{2}r\rho$, we find

$$r_c = -2W'(\rho_c).$$

To find the exponent β we need the relation between the r and ρ near the critical point. In a generic situation we find

$$t = r - r_c \sim -2W''(\rho_c)(\rho - \rho_c) \;\Rightarrow\; \beta = \tfrac{1}{2}\,.$$

This is the case of a simple critical point that we have examined in Chapter 30. A new situation arises if one can adjust another parameter of the potential in such a way that $W''(\rho_c) = 0$. This provides an example of a tricritical point. It can be achieved only if the potential U is at least cubic. Higher critical points can be obtained when more derivatives vanish. However this is no longer interesting for continuous symmetries since for $d \geq 3$ mean-field behaviour then is valid (and symmetry breaking is possible only for $d > 2$).

At high temperature $\sigma = 0$ and m, which is the σ-field mass, is given by

$$2W'(\rho) + r = m^2\,, \qquad \rho - \rho_c = -\frac{1}{(2\pi)^d}m^2\int\frac{\mathrm{d}^d p}{p^2(p^2+m^2)}\,.$$

For $d < 4$ we find $\rho - \rho_c \sim m^{d-2}$. In the tricritical case then

$$m^2 = t + \text{const.} \times W'''(\rho_c)m^{2(d-2)}.$$

For $d = 3$ we find a mean field behaviour without logarithmic corrections. We recall that $d = 3$ is the upper-critical dimension.

Discussion. In the mean field approximation the function $W(\rho) = \rho^3$ is not bounded from below, however $\rho = 0$ is the minimum because by definition $\rho \geq 0$. Here instead we are in the situation where $W(\rho) \sim (\rho - \rho_c)^3$ but ρ_c is positive. Thus this extremum of the potential is unstable. The tricritical point and the $(\phi^2)^3$ field theory cannot be studied systematically by a large N expansion.

44.25 Two-dimensional Models and Bosonization Method : Chapter 32

Exercise 25.1

Study in the 2D free massless scalar theory the short distance expansion of the product of operators $\mathrm{e}^{i\kappa_1\varphi(x_1)}\,\mathrm{e}^{i\kappa_2\varphi(x_2)}$.

Solution. Distinguishing two operators in equation (32.13) we find

$$\left.\left\langle\prod_{i=1}^{n}\mathrm{e}^{i\kappa_i\varphi(x_i)}\right\rangle\right|_{\text{ren.}}=(\mu\,|x_1-x_2|)^{\kappa_1\kappa_2/2\pi}\prod_{i>2}(\mu\,|x_1-x_i|)^{\kappa_i\kappa_1/2\pi}\times(\mu\,|x_2-x_i|)^{\kappa_i\kappa_2/2\pi}\prod_{2<i<j}(\mu\,|x_i-x_j|)^{\kappa_i\kappa_j/2\pi}.$$

Taking the limit $x_2\to x_1$ we see that the first factor is the singular coefficient function and thus

$$\mathrm{e}^{i\kappa_1\varphi(x_1)}\,\mathrm{e}^{i\kappa_2\varphi(x_2)}\underset{x_2\to x_1}{\propto}C_{\kappa_1,\kappa_2}(x_1-x_2)\,\mathrm{e}^{i(\kappa_1+\kappa_2)\varphi(x_1)},$$

with

$$C_{\kappa_1,\kappa_2}(x)\sim x^{\kappa_1\kappa_2/2\pi}.$$

Moreover the behaviour of the coefficient function C is related in a way expected from the discussion of Section 12.7 to the dimensions of the three operators (see equation (32.16)) since

$$\kappa_1\kappa_2/2\pi=(\kappa_1+\kappa_2)^2/4\pi-\kappa_1^2/4\pi-\kappa_2^2/4\pi=[\mathrm{e}^{i(\kappa_1+\kappa_2)\varphi}]-[\mathrm{e}^{i\kappa_1\varphi}]-[\mathrm{e}^{i\kappa_2\varphi}].$$

Exercise 25.2

Consider 2D QED in the case of N identical charged massless Dirac fermions. Denoting the fermion fields by $\boldsymbol{\psi},\bar{\boldsymbol{\psi}}$ we write the action

$$\mathcal{S}\left(\bar{\boldsymbol{\psi}},\boldsymbol{\psi},A_\mu\right)=\int\mathrm{d}^2x\left[\tfrac{1}{4}F_{\mu\nu}^2(x)-\bar{\boldsymbol{\psi}}(x)\cdot(\not{\partial}+ie\not{A})\,\boldsymbol{\psi}(x)\right].\tag{44.78}$$

Bosonize the model and identify the symmetry breaking operator with non-zero expectation value.

Solution. The model (44.78) possesses, in addition to gauge invariance, a $U(N)\times U(N)$ chiral symmetry because the fermions are massless. The model is a simple extension of the Schwinger model and can be solved exactly. As we shall see, however, for higher values of N its structure is slightly more complicated.

The first step is the evaluation of anomaly. This is simple because the fermion determinant is replaced by the determinant to the power N. The generalization of the free action (32.61) thus is

$$\mathcal{S}(\bar{\boldsymbol{\psi}}',\boldsymbol{\psi}',B_\mu)=-\int\mathrm{d}^2x\left[\bar{\boldsymbol{\psi}}'(x)\cdot\not{\partial}\boldsymbol{\psi}'(x)+\frac{N}{2\pi}\int\mathrm{d}^2x\big(\partial_\mu\varphi(x)\big)^2\right].\tag{44.79}$$

Adding the gauge field free action which is unchanged we obtain

$$\mathcal{S}(\bar{\boldsymbol{\psi}}',\boldsymbol{\psi}',\varphi)=-\int\mathrm{d}^2x\left[\bar{\boldsymbol{\psi}}'\cdot\not{\partial}\boldsymbol{\psi}'+\frac{1}{2e^2}\partial_\mu\varphi\left(m^2-\nabla^2\right)\partial_\mu\varphi\right],\tag{44.80}$$

with

$$m^2 = Ne^2/\pi\,. \tag{44.81}$$

The φ propagator has the form:

$$\langle\varphi(x)\varphi(0)\rangle = \frac{1}{4\pi^2}\int \mathrm{d}^2p\ \mathrm{e}^{ipx}\,\frac{\pi}{N}\left(\frac{1}{p^2+m^2}-\frac{1}{p^2}\right), \tag{44.82}$$

and therefore propagates a neutral massive state together with a negative metric massless state, as in the Schwinger model. Since this boson is free it cannot produce charged fermions and this is an indication of confinement.

Let us now bosonize the fermion fields $\bar\psi', \psi'$, introducing N scalar bosons ϑ_i. The equivalent boson action is

$$\mathcal{S}(\vartheta_i,\varphi) = \tfrac{1}{2}\int \mathrm{d}^2x\left[\partial_\mu\vartheta_i\partial_\mu\vartheta_i - \frac{1}{e^2}\partial_\mu\varphi\left(m^2-\partial^2\right)\partial_\mu\varphi\right]. \tag{44.83}$$

Then we proceed as for the Schwinger model, translating $\vartheta_i + \varphi/\sqrt{\pi} \mapsto \vartheta_i$ and integrate over φ. The action takes the form

$$\mathcal{S}(\vartheta_i) = \tfrac{1}{2}\int \mathrm{d}^2x\left[\partial_\mu\vartheta_i\partial_\mu\vartheta_i + \frac{m^2}{N}\left(\textstyle\sum_i\vartheta_i\right)^2\right]. \tag{44.84}$$

Through all these transformations the chiral components $\sigma^i_\pm$ of $\bar\psi_i\psi_i$ become simply

$$\sigma^i_\pm(x) = \bar\psi^i_\mp(x)\psi^i_\pm(x) \ \longmapsto\ \frac{\Lambda}{2\pi}\,\mathrm{e}^{\pm i\sqrt{4\pi}\vartheta_i(x)}\,. \tag{44.85}$$

The first remark is that in contrast to the Schwinger model, besides one massive field $\sum_i\vartheta_i$, we have in addition $N-1$ massless boson fields.

The $\sigma^i_\pm$ correlation functions. Due to the presence for $N>1$ of massless bosons exxpectation values of the form $\angle\sigma^i_\pm\rangle$ all vanish. However let us calculate the average of the operator

$$\left\langle\prod_{i=1}^N \sigma^i_+(x)\right\rangle = \frac{\Lambda^N}{(2\pi)^N}\left\langle \mathrm{e}^{i\sqrt{4\pi}\sum_i\vartheta_i(x)}\right\rangle = \frac{\Lambda^N}{(2\pi)^N}\,\mathrm{e}^{-2N\pi\Delta(0,m)} = \left(\langle\sigma\rangle_{N=1}\right)^N.$$

In the fermion representation the operator is given by

$$\prod_{i=1}^N \sigma^i_+(x) = (-1)^{N(N-1)/2}\prod_{i=1}^N\bar\psi^i_-(x)\prod_{i=1}^N\psi^i_+(x).$$

Due to the antisymmetry of the fermions, if we perform a $U(N)\times U(N)$ transformation of matrices U_+, U_-, the operator transforms by multiplication by $\det U_+/\det U_-$. The operator is thus invariant under the $SU(N)\times SU(N)\times U(1)$ group. Its non-vanishing expectation value breaks only the remaining $U(1)$ chiral component.

Finally the confirmation of confinement is here a more complicated problem since spectrum considerations are not sufficient.

Exercise 25.3

To verify the conformal invariance of the free massless boson field theory explicitly calculate the two-point function $\left\langle e^{ik_2\phi(x,y)}\, e^{ik_1\phi(0,0)}\right\rangle$ in the cylindrical geometry, i.e. with a finite size L and periodic boundary conditions in one direction (see also Section 37.5 for the problem of the zero mode).

Solution. We first expand the field $\phi(x,y)$ in Fourier modes:

$$\phi(x,y)=\sum_n e^{2i\pi ny/L}\,\phi_n(x).$$

The corresponding action takes the form

$$\mathcal{S}(\phi)=\tfrac{1}{2}L\int dx\sum_n\left(\phi'_n(x)\phi'_{-n}(x)+(4\pi^2n^2/L^2)\phi_n(x)\phi_{-n}(x)\right).$$

As usual in the finite size problem we have to consider separately the zero momentum mode which would naively lead to IR divergences. The zero mode corresponds to a free particle in quantum mechanics. A simple calculation shows that its contribution to the two-point function is a factor

$$(2\pi)\delta(k_1+k_2)\,e^{-\frac{1}{2}k_1^2|x|/L}.$$

The δ-function is expected from the analysis of Section 32.1. The propagator for the other modes then takes the form

$$\begin{aligned}\Delta_L(x,y)&=\frac{1}{2\pi L}\sum_{n\neq 0}e^{2i\pi ny/L}\int d\omega\,\frac{e^{i\omega x}}{\omega^2+(2\pi n/L)^2},\\&=\frac{1}{4\pi}\sum_{n>0}\frac{1}{n}\,e^{-2\pi n|x|/L}\left(e^{2i\pi ny/L}+e^{-2i\pi ny/L}\right)\\&=-\frac{1}{4\pi}\ln\left[\left(1-e^{-2\pi|x|/L}\,e^{2i\pi y/L}\right)\left(1-e^{-2\pi|x|/L}\,e^{-2i\pi y/L}\right)\right].\end{aligned}$$

The propagator at coinciding points is UV divergent:

$$\Delta_L(0,0)=\frac{1}{2\pi}\ln(L\Lambda)+O(1).$$

Performing the gaussian integration we obtain the two-point function

$$\begin{aligned}\left\langle e^{ik_2\phi(x,y)}\, e^{ik_1\phi(0,0)}\right\rangle&\propto\delta(k_1+k_2)\,e^{-\frac{1}{2}k_1^2|x|/L}\,e^{-k_1^2\ln(L\Lambda)/2\pi}\,e^{k_1^2\Delta_L(x,y)}\\&\propto\delta(k_1+k_2)\left[\Lambda^2L^2\sinh\left(\frac{\pi w}{L}\right)\sinh\left(\frac{\pi\bar{w}}{L}\right)\right]^{-k_1^2/(4\pi)},\end{aligned}$$

where we have set $w=x+iy$, and $\bar{w}$ is the complex conjugate of w.

We now observe that the conformal transformation $z=e^{2\pi w/L}$, maps the complex plane onto a periodic strip of size L. The propagator in the initial z variable is

$$\Delta(z,\bar{z})=-\frac{1}{4\pi}\ln z\bar{z}\,.$$

Then the propagator in the new variables between the points $w = x + iy$ and $w = 0$ has the form

$$\Delta = -\frac{1}{4\pi} \ln \left(1 - \mathrm{e}^{2\pi w/L}\right) \left(1 - \mathrm{e}^{2\pi \bar{w}/L}\right),$$

which coincides with the result obtained by direct calculation. One can actually verify that in a conformal mapping $z \mapsto w$ the two-point function of $\mathrm{e}^{ik\phi(z)}$ transforms by the substitution $z \mapsto w$ and is multiplied by a factor coming the dimension $k_1^2/(8\pi)$ of the operator

$$\left(\frac{\partial z}{\partial w}\right)^{-k^2/8\pi} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{-k^2/8\pi}.$$

Exercise 25.4

Find the most general analytic function $f(z)$ which satisfies an identity of the form (32.33)

$$(-1)^{n+1} \det \frac{1}{f(z_i - z'_j)} = \frac{\prod_{i<j} f(z_i - z_j) f(z'_i - z'_j)}{\prod_{i,j} f(z_i - z'_j)}. \tag{44.86}$$

One assumes $f'(0) \neq 0$.

Solution. Let us first determine the function $f(z)$ by exploiting the identity for $n = 2$. Multiplying by the denominators one finds

$$f(z_1 - z'_1) f(z_2 - z'_2) - f(z_1 - z'_2) f(z_2 - z'_1) = f(z_1 - z_2) f(z'_1 - z'_2).$$

Setting $z_1 = z_2 = 0$ one finds $f(0) = 0$. Setting then $z_2 = z'_2$ one concludes that $f(z)$ is an odd function. One finally differentiates twice with respect to z'_1 and once with respect to z'_2. One sets $z_2 = z'_1 = z'_2 = 0$. A differential equation for f follows

$$f''(z) f'(0) - f(z) f'''(0) = 0\,.$$

If $f'''(0)$ vanishes we recover the linear function we have considered so far. However we have another solution

$$f(z) = \sinh(\alpha z).$$

It is then easy to verify that the function satisfies the identity for all n by noting

$$\frac{1}{\sinh\bigl(\alpha(z - z')\bigr)} = \frac{2\,\mathrm{e}^{\alpha z}\,\mathrm{e}^{\alpha z'}}{\mathrm{e}^{2\alpha z} - \mathrm{e}^{2\alpha z'}}.$$

We can factorize the numerators in front of the determinant, and in the denominators we recognize the Cauchy determinant with variables $Z = \mathrm{e}^{2\alpha z}$. Therefore we can also bosonize a system in a strip or at finite temperature.

44.26 Critical Properties of Gauge Theories : Chapter 34

The Landau–Ginzburg model of superconductivity. The Landau–Ginzburg model of a superconductor is, in field theory language, the abelian Higgs model, i.e. a field theory for a charged scalar field coupled to an electromagnetic gauge field (for more details see the classical books on superconductivity)

$$\mathcal{S}(A_\mu, \phi) = \int \mathrm{d}^d x \left(\frac{1}{4}F_{\mu\nu}^2 + |\mathrm{D}_\mu \phi|^2 + M^2 |\phi|^2 + \frac{1}{6} g |\phi|^4\right),$$

with:

$$D_\mu = \partial_\mu + ieA_\mu \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

We propose to study by RG methods the nature of the phase transition. Calculate the RG β-functions at one-loop order, solve the flow equations for the coupling constants in 4 and in $4-\varepsilon$ dimensions. Discuss the RG flow, the locations and stability of the fixed points.

A useful reference is:

S. Coleman and E. Weinberg, *Phys. Rev.* D7 (1973) 1888.

Solution. Introducing the parameters $\tilde{g}, \tilde{\alpha}$

$$\tilde{g} = g/8\pi^2 \,, \qquad \tilde{\alpha} = e^2/8\pi^2,$$

we obtain for the corresponding β-functions (see equation (18.134) for $N = 1$)

$$\beta_{\tilde{g}} = -\varepsilon \tilde{g} + \tfrac{5}{3}\tilde{g}^2 - 6\tilde{g}\tilde{\alpha} + 18\tilde{\alpha}^2 \,, \qquad (44.87a)$$

$$\beta_{\tilde{\alpha}} = -\varepsilon \tilde{\alpha} + \tfrac{1}{3}\tilde{\alpha}^2. \qquad (44.87b)$$

Setting $t = \ln \lambda$, where λ is the scale parameter, and $\tilde{g} = \tilde{\alpha} s$, we can write the flow equations at $d = 4$:

$$\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} = \tfrac{1}{3}\tilde{\alpha}^2,$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \tilde{\alpha}\left(\tfrac{5}{3}s^2 - \tfrac{19}{3}s + 18\right).$$

The first gives the flow of $\tilde{\alpha}$ which goes to zero at large distance ($t \to -\infty$), $\tilde{\alpha} \sim -3/t$. One verifies that the factor in the r.h.s. of (44.87a) never vanishes. Therefore $s(t)$ decreases at large distance and passes through the origin for a finite t. Since stability implies that $s(t)$ should remain positive, we face a case of runaway coupling constant, a strong indication of a first order phase transition.

One then verifies that for the same reason in $d = 4 - \varepsilon$ there is no solution to the IR fixed point equations and again one predicts a first order phase transition. This first order character is not observed experimentally, the belief being that the critical domain is too small.

Finally by increasing enough the number N of charged scalar fields eventually one reaches a situation where the transition becomes second order, a result which can also be checked in the large N limit.

44.27 Instantons in Quantum Mechanics : Chapter 39

Exercise 27.1

Calculate the imaginary part of the ground state energy for the hamiltonian

$$H = -\tfrac{1}{2}\left(\mathrm{d}/\mathrm{d}q\right)^2 + \tfrac{1}{2}q^2 + \tfrac{1}{2}gq^{2N},$$

for g negative.

A result useful for all exercises: if $M(t_1,t_2)$ is of the form:

$$M(t_1,t_2) = \left[-\left(\frac{\mathrm{d}}{\mathrm{d}t_1}\right)^2 + 1 - \frac{\lambda(\lambda+1)\,\omega^2}{\cosh^2 \omega t_1}\right]\delta(t_1 - t_2),$$

in the notation of equation (39.36) we have:

$$\det(M+\varepsilon)(M_0+\varepsilon)^{-1} = \frac{\Gamma(1+z)\,\Gamma(z)}{\Gamma(1+\lambda+z)\,\Gamma(z-\lambda)}, \tag{44.88}$$

with

$$z = \sqrt{1+\varepsilon}/\omega\,.$$

Solution. A few elements of the solution are:

(i) The classical solution

$$|q_c|^{N-1}(t) = \frac{1}{\sqrt{-g}}\frac{1}{\cosh[(N-1)(t-t_0)]}.$$

(ii) The classical action $\mathcal{S}_c$

$$\mathcal{S}_c = A(N)/(-g)^{1/(N-1)},$$

$$A(N) = \frac{\sqrt{\pi}\,\Gamma\big(N/(N-1)\big)}{2\Gamma\big((3N-1)/2(N-1)\big)} = 4^{1/(N-1)}\frac{\Gamma^2\big(N/(N-1)\big)}{\Gamma\big(2N/(N-1)\big)}.$$

(iii) The operator M, second functional of the action at the saddle point, is

$$M = -\mathrm{d}_t^2 + 1 - \frac{N(2N-1)}{\cosh^2[(N-1)t]}.$$

Then the determinant is given by equation (44.88) with

$$\lambda = N/(N-1)\,, \qquad z = \sqrt{1+\epsilon}/(N-1).$$

It follows

$$\det(M+\varepsilon)(M_0+\varepsilon)^{-1} \sim -2^{-(N+1)/(N-1)}A(N)\varepsilon\,.$$

(iv) The operator M again has one zero eigenvalue associated with time translation. The jacobian J induced by taking time as a collective coordinate is

$$J^2 = \int \mathrm{d}t\,\dot{q}^2(t) = A/(-g)^{1/(N-1)}.$$

Using equation (39.35) and collecting all factors, one obtains the imaginary part of the ground state energy for negative g:

$$\operatorname{Im} E(g) = C(-g)^{-\beta}\exp\left[-A(N)/(-g)^{1/(N-1)}\right], \tag{44.89}$$

with

$$\beta = 1/[2(N-1)], \qquad C = \frac{2^{1/(N-1)}}{\sqrt{\pi}}.$$

Exercise 27.2

Calculate the imaginary part of the ground state energy for the hamiltonian with $O(\nu)$ internal symmetry

$$H = -\tfrac{1}{2}\left(\mathrm{d}/\mathrm{d}\mathbf{q}\right)^2 + \tfrac{1}{2}\mathbf{q}^2 + \tfrac{1}{2}g\left(\mathbf{q}^2\right)^N.$$

Solution. We only explain the main new features of the solution, compared to the $\nu = 1$ case. The classical solutions are

$$\mathbf{q}_c(t) = \mathbf{n}\, q_c(\nu = 1, t),$$

where $\mathbf{n}$ is an arbitrary constant unit vector. Therefore the classical action is the same as in the $\nu = 1$ case. The main difference comes from the degeneracy of the saddle point: We have to include the unit vector $\mathbf{n}$ in our set of collective coordinates. This leads to a new jacobian and to a factor which is the surface of the ν dimensional sphere.

The second derivative of the action is

$$M_{ij} = \left(-\mathrm{d}_t^2 + 1\right)\delta_{ij} - N\bigl(\delta_{ij} + 2(N-1)n_i n_j\bigr)\frac{1}{\cosh^2[(N-1)t]}.$$

The determinant can be expressed in terms of the determinants of a transverse and a longitudinal operator (with respect to the vector $\mathbf{n}$)

$$\begin{aligned} M_{\mathrm{L}} &= -\mathrm{d}_t^2 + 1 - \frac{N(2N-1)}{\cosh^2[(N-1)t]}\,, \\ M_{\mathrm{T}} &= -\mathrm{d}_t^2 + 1 - \frac{N}{\cosh^2[(N-1)t]}\,. \end{aligned}$$

The longitudinal determinant is the same as before. Its zero mode corresponds to time translation. The transverse determinant which appears $\nu - 1$ times corresponds to $\lambda = 1/(N-1)$ and its zero mode is related to rotations of the vector $\mathbf{n}$:

$$\det\left(M_{\mathrm{T}} + \varepsilon\right)\left(M_0 + \varepsilon\right)^{-1} \sim \frac{N+1}{N-1} 2^{-(N+1)/(N-1)} A(N)\varepsilon\,.$$

Note that all eigenvalues of M_{T} are positive, in such a way that the final result remains real as expected.

The eigenvectors corresponding to a rotation are all unit vectors orthogonal to $\mathbf{n}$ multiplied by $q_c(\nu = 1, t)$. The corresponding jacobian thus is

$$J_{\mathrm{T}} = \left(\int \mathrm{d}t\, q_c^2(t)\right)^{(\nu-1)/2} = \left(\frac{N+1}{N-1}A\right)^{(\nu-1)/2} \frac{1}{(-g)^{(\nu-1)/2(N-1)}}.$$

We have still to divide by a factor $\sqrt{2\pi}$ for each zero-mode. Finally the integration over $\mathbf{n}$ yields the surface of the sphere. Taking also into account the factor $1/2i$ to obtain the imaginary part we find an additional factor:

$$\frac{1}{2i}\frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}\frac{1}{(2\pi)^{\nu/2}}\,.$$

Collecting all factors we finally obtain, in the notation (44.89):

$$\beta = \nu/[2(N-1)], \qquad C = \frac{1}{\Gamma(\nu/2)} 2^{\nu/(N-1)}\,.$$

Exercise 27.3

Use the shifting method, rather than the method of Section 2.3, to calculate the path integral giving the matrix elements of the euclidean evolution operator $U(x',x;\beta)$ for the simple harmonic oscillator

$$U(x',x;\beta)=\int_{r(0)=x}^{r(\beta)=x'}[\mathrm{d}r(t)]\exp\left[-\mathcal{S}\big(r(t)\big)\right],$$
$$\mathcal{S}(r)=\tfrac{1}{2}\int_0^\beta \mathrm{d}t\left(\dot r^2(t)+\omega^2 r^2(t)\right).$$

Solution. The action $\mathcal{S}(r)$ can be rewritten

$$\mathcal{S}(r)=\tfrac{1}{2}\int_0^\beta \mathrm{d}t\big(\dot r(t)+\omega r(t)\big)^2+\tfrac{1}{2}\omega\left(x^2-x'^2\right).$$

Note that this identity remains valid within the path integral only if we set $\epsilon(0)=0$, where $\epsilon(t)$ is the sign function.

We then change variables ($\kappa(t)=\mathrm{e}^{-\omega t}$)

$$r(t)=\sigma(t)-\omega\,\mathrm{e}^{-\omega t}\int_0^t \mathrm{d}\tau\,\sigma(\tau)\,\mathrm{e}^{\omega\tau}=\mathrm{e}^{-\omega t}\left(x+\int_0^t\mathrm{d}\tau\,\dot\sigma(\tau)\,\mathrm{e}^{\omega\tau}\right),$$

which corresponds to the boundary condition $\sigma(0)=x$.

The jacobian J of the transformation is (equation (39.70))

$$J=\mathrm{e}^{-\omega\beta/2}.$$

We have still to impose the condition $r(\beta)=x'$. We introduce a variable λ to write the corresponding δ-function in exponential form. We obtain the action

$$\mathcal{S}(\sigma,\lambda)=\tfrac{1}{2}\int_0^\beta \mathrm{d}t\,\dot\sigma^2(t)+i\lambda\left[x'-x\,\mathrm{e}^{-\omega\beta}-\mathrm{e}^{-\omega\beta}\int_0^\beta\mathrm{d}t\;\mathrm{e}^{\omega t}\,\dot\sigma(t)\right]+\tfrac{1}{2}\omega\left(x^2-x'^2\right).$$

Shifting then $\sigma(t)$

$$\dot\sigma(t)=\dot\sigma'(t)+i\lambda\,\mathrm{e}^{\omega(t-\beta)},\quad \sigma'(0)=x,$$

we find

$$\mathcal{S}(\sigma,\lambda)=\tfrac{1}{2}\int_0^\beta\mathrm{d}t\,\dot\sigma^2(t)+\frac{\lambda^2}{4\omega}\left(1-\mathrm{e}^{-2\omega\beta}\right)+i\lambda\left(x'-x\,\mathrm{e}^{-\omega\beta}\right)+\tfrac{1}{2}\omega\left(x^2-x'^2\right).$$

The integral over $\sigma(t)$ now is

$$\int_{\sigma(0)=x}[\mathrm{d}\sigma(t)]\exp\left[-\tfrac{1}{2}\int_0^\beta\mathrm{d}t\,\dot\sigma^2(t)\right]=\frac{1}{\sqrt{2\pi\beta}}\int\mathrm{d}s\;\exp[-\tfrac{1}{2}(x-s)^2/\beta]=1.$$

The integral over λ then leads to

$$U(x',x;\beta)=\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\omega}}{\sqrt{\sinh\omega\beta}}\exp[-\tfrac{1}{2}\omega A(x,x')],$$

with

$$A(x,x')=x^2-x'^2+2\frac{\left(x'-x\,\mathrm{e}^{-\omega\beta}\right)^2}{1-\mathrm{e}^{-2\omega\beta}}=\frac{(x^2+x'^2)\cosh\omega\beta-2xx'}{\sinh\omega\beta},$$

which is the correct result. The factor $(2\pi)^{-1/2}$ is the product of a factor $1/(2\pi)$ from the representation of the δ-function by the factor $(2\pi)^{1/2}$ of the gaussian integration.

Exercise 27.4

Calculate the imaginary part of the Nth level above the ground state at leading order.

Solution. All the relevant equations can already be found in Section 39.7. To calculate the imaginary part of the Nth level we have to identify the coefficient of $\mathrm{e}^{-(N+1/2)\beta}$ in the large β expansion of the partition function. The leading contribution then comes from the term with the largest power of $1/g$. If we expand $E(\beta)$ and $A(\beta)$ in (39.91) we observe that the leading contribution comes from the correction of order $\mathrm{e}^{-\beta}$ in $A(\beta)$. From equation (39.101)

$$A(\beta) = a - 2C\,\mathrm{e}^{-\beta} + O\left(\mathrm{e}^{-2\beta}\right),$$

we then find

$$\operatorname{Im} E_N(g) \sim \frac{1}{N!}\left(\frac{2C}{g}\right)^N \operatorname{Im} E_0(g).$$

We notice that, as expected, the imaginary part increases with N and thus the lifetime decreases. However this is the result for $g \to 0$ at N fixed. For N large enough a new regime becomes relevant, the WKB regime which corresponds to $g \to$, $N \to \infty$ at fixed energy.

27.4.1. Calculate the imaginary part of the ground state energy for a system with $O(\nu)$ internal symmetry, with the action

$$\mathcal{S}(\mathbf{q}) = \int \mathrm{d}t \left[\tfrac{1}{2}\dot{\mathbf{q}}^2 + \mathcal{V}\left(g\mathbf{q}^2\right)/g\right].$$

Solution. Barrier penetration effects are related to euclidean trajectories with go from the origin to a turning point and bounce back. If the action is $O(\nu)$ invariant such trajectories will take the form of the $O(1)$ trajectories multiplied by a constant unit vector $\mathbf{n}$:

$$\mathbf{q}_c(t) = \mathbf{n}\, q_c(t).$$

Therefore the classical action is independent of ν and the analysis of the $\nu = 1$ can be repeated. The main difference comes from the gaussian integration. Note first that it is simpler to calculate the contribution to $\operatorname{tr} \mathrm{e}^{-\beta H}$ than to the matrix elements, because the classical solutions are simpler. The classical solution, being periodic, is determined by one boundary condition for example the value $\mathbf{x} = \mathbf{q}(-\beta/2)$. The variable $\mathbf{x}$ plays the role of collective coordinates corresponding to time translation and direction in $\mathbf{q}$ space.

We have then to find a non-singular matrix K satisfying equation (39.86). The equation for K takes the form

$$\ddot{K}_{ij} = 2\mathcal{V}'\left(g\mathbf{q}^2\right) K_{ij} + 4g n_i q_c^2(t) \mathcal{V}\left(g q_c^2(t)\right) n_k K_{kj}\,.$$

We have to find enough independent solutions. Here the problem is rather simple. We decompose the matrix $\mathbf{K}$ as

$$K_{ij} = K_{\mathrm{L}} n_i n_j + \left(\delta_{ij} - n_i n_j\right) K_{\mathrm{T}}\,.$$

For what concerns K_{L} one solution is $\dot{q}_c(t)$ and the problem is the same as in the $\nu = 1$ case. For K_{T} we recognize the equation of motion and therefore a solution is $q_c(t)$ itself. The calculation then is an extension of the $\nu = 1$ example and the result is, with the notation (39.100):

$$\operatorname{Im} E_0(g) \sim \frac{1}{\Gamma(\nu/2)}\left(-\frac{2C}{g}\right)^{\nu/2} \mathrm{e}^{a/g}\,.$$

Exercise 27.5

Establish, using the results of Section A39.2, the semiclassical expression of the scattering amplitude for a $O(d)$ invariant hamiltonian with a radial potential $V(|\mathbf{x}|)$:

$$H = \tfrac{1}{2}\mathbf{p}^2 + V(r), \quad r = |\mathbf{x}|.$$

Solution. As we have shown in Section 6.1 we have to evaluate the evolution operator with classical scattering boundary conditions. We recall the expression of the evolution operator at leading WKB order (Van Vleck's formula for $\hbar = 1$), as given in Section A39.2:

$$\langle \mathbf{x}'' \left| U\left(T'', T'\right) \right| \mathbf{x}' \rangle \sim \frac{1}{\left(2\pi i\right)^{n/2}} \left(-\det \frac{\partial^2 \mathcal{A}}{\partial \mathbf{x}' \partial \mathbf{x}''}\right)^{1/2} \exp\left[i\mathcal{A}\left(\mathbf{x}'', \mathbf{x}'; T'', T'\right)\right],$$

where $\mathcal{A}$ is the classical action evaluated for the solutions of the equation of motion.

The classical equation of motions, after integration, yield

$$\tfrac{1}{2}\dot{\mathbf{x}}^2 + V(r) = \tfrac{1}{2}\kappa^2, \qquad r^2\dot{\theta} = l\,,$$

where κ, l are two constants of the motion and θ characterizes the angular position of the vector $\mathbf{x}$ in the plane of the motion. Taking as boundary conditions $\mathbf{x}(t') = \mathbf{x}'$, $\mathbf{x}(t'') = \mathbf{x}''$, setting $T = t'' - t'$ and

$$\kappa(r) = \left[\kappa^2 - 2V(r) - \frac{l^2}{r^2}\right]^{1/2},$$

we can write the two equations determining κ, l

$$\int_L \frac{\mathrm{d}r}{\kappa(r)} = T\,, \qquad l\int_L \frac{\mathrm{d}r}{r^2\kappa(r)} = \alpha\,,$$

where α is the angle between $\mathbf{x}'$ and $\mathbf{x}''$ and L means the domain of integration

$$\int_L \equiv \int_{r_0}^{r'} + \int_{r_0}^{r''}, \qquad \kappa(r_0) = 0\,.$$

Finally the classical action $\mathcal{A}$ is given by

$$\mathcal{A} = \int \mathrm{d}t\left[\tfrac{1}{2}\dot{\mathbf{x}}^2 - V(r)\right] = \tfrac{1}{2}T\kappa^2 - 2\int_L \frac{V(r)\mathrm{d}r}{\kappa(r)} = \tfrac{1}{2}T\kappa^2 + l\alpha + \int_L \mathrm{d}r\left(\kappa(r) - \frac{\kappa^2}{\kappa(r)}\right).$$

It is useful to to calculate the variation of the action when the end-points vary

$$\delta\mathcal{A} = l\delta\alpha + \kappa(r'')\delta r'' - \kappa(r')\delta r'.$$

As a consequence:

$$\begin{aligned}
\frac{\partial \mathcal{A}}{\partial x'_i} &= \frac{l}{r'\sin\alpha}\left(\cos\alpha\frac{x'_i}{r'} - \frac{x''_i}{r''}\right) - \kappa(r')\frac{x'_i}{r'},\\
\frac{\partial \mathcal{A}}{\partial x''_i} &= \frac{l}{r''\sin\alpha}\left(\cos\alpha\frac{x''_i}{r''} - \frac{x'_i}{r'}\right) + \kappa(r'')\frac{x''_i}{r''}.
\end{aligned}$$

A few other formula are useful:

$$\frac{\partial^2 \alpha}{\partial x'_i \partial x''_j} = -\frac{1}{r' r'' \sin\alpha}\left[\delta_{ij} - \frac{x'_i x'_j}{r'^2 \sin^2\alpha} - \frac{x''_i x''_j}{r''^2 \sin^2\alpha} + \frac{(x'_i x''_j + x''_i x'_j)\cos\alpha}{r' r'' \sin^2\alpha}\right].$$

We note that the result is a projector orthogonal to $\mathbf{x}'$ and $\mathbf{x}''$. Finally we can derive

$$\det \frac{\partial^2 \mathcal{A}}{\partial x'_i \partial x''_j} = \frac{1}{r' r''}\left(\frac{-\partial \mathcal{A}}{\partial \alpha}\frac{1}{r' r'' \sin\alpha}\right)^{d-2}\left(\frac{\partial^2 \mathcal{A}}{\partial\alpha\partial r'}\frac{\partial^2 \mathcal{A}}{\partial\alpha\partial r''} - \frac{\partial^2 \mathcal{A}}{\partial\alpha\partial\alpha}\frac{\partial^2 \mathcal{A}}{\partial r'\partial r''}\right).$$

A short calculation then yields

$$\det \frac{\partial^2 \mathcal{A}}{\partial x'_i \partial x''_j} = \left(-\frac{l}{r' r'' \sin\alpha}\right)^{d-2} \frac{1}{r' r'' \kappa(r')\kappa(r'') I}\frac{\partial l}{\partial \alpha},$$

with

$$I = \frac{-1}{\kappa}\frac{\partial}{\partial \kappa}\bigg|_l \int_L \frac{\mathrm{d}r}{\kappa(r)}.$$

Large time expansion. Knowing from the general analysis that the boundary conditions should correspond to classical scattering, we can expand for T large (we assume that $V(r)$ decreases fast enough). Setting

$$k = (r' + r'' - 2r_0)/T, \qquad k(r) = \left[k^2 - V(r) - \frac{l^2}{r^2}\right]^{1/2},$$

where r_0 is replaced by the limiting value solution of $k(r_0) = 0$ we find

$$\frac{kT}{\kappa} + 2\int_{r_0}^{\infty} \mathrm{d}r \left(\frac{1}{k(r)} - \frac{1}{k}\right) + O\left(T^{-1}\right) = T,$$

and thus

$$\kappa = k + \frac{2}{T}\int_{r_0}^{\infty} \mathrm{d}r \left(\frac{k}{k(r)} - 1\right) + O\left(T^{-2}\right).$$

It follows after a short calculation

$$\alpha = 2l \int_{r_0}^{\infty} \frac{\mathrm{d}r}{r^2 k(r)},$$

$$\mathcal{A} = \tfrac{1}{2}Tk^2 + 2\int_{r_0}^{\infty} \mathrm{d}r\,[k(r) - k] + l\alpha.$$

In this limit l is related to the impact parameter b by $b = l/k$. We then calculate the Fourier transform of the evolution operator. At leading order for T large the action depends only on $r' + r''$. We can thus integrate over the angles. We find Bessel's functions with large arguments $p'r'$ and $p''r''$ which can be replaced by their asymptotic forms. These correspond to the angle between $\mathbf{x}$ and $\mathbf{p}$ being 0 or π. The integral over $r' - r''$ then yields the energy conservation δ-function, $p'^2 = p''^2$. The integral over $r' + r''$ then implies $k = p'$. Thus finally we can substitute

$$p'^2 = p''^2 = k^2, \qquad k^2 \cos\alpha = -\mathbf{p}' \cdot \mathbf{p}''.$$

All these terms give only a phase ϕ obtained by adding to the result the contribution $\frac{1}{2}p'^2$ coming from the free terms

$$\phi = 2\int_{r_0}^{\infty} \mathrm{d}r\,[k(r) - k] + l\alpha - 2ikr_0\,,$$

while the amplitude is completely given by the determinant. Thus the final result for the cross-section is proportional to

$$\frac{\partial\sigma(k,\alpha)}{\partial\alpha} \propto \frac{\partial b}{\partial\alpha} b^{d-2},$$

a result that can also be obtained directly by a purely classical calculation.

44.28 Unstable Vacua in Field Theory : Chapter 40

Exercise 28.1

Find an estimate for the instanton action in the ϕ^4 field theory in $d = 2, 3, 4$ dimensions by using a first gaussian trial function, then a function proportional to the one-dimensional solution, $f(r) \propto 1/\cosh(r/r_0)$, and compare with the exact results given in Section 29.3, for $d = 2, 3$ and in Section 40.3 for $d = 4$.

Solution. The expression ($A40.27$) is independent of the normalization of φ and the scale of distances. Therefore we just substitute

$$\varphi(r) = \mathrm{e}^{-r^2/2},$$

and find $A(d) = 24\pi^{d/2}(4-d)^{(d-4)/2}$ and thus

$$A(2) = 37.699\ldots\,, \quad A(3) = 133.639\ldots\,, \quad A(4) = 24\pi^2\ .$$

For the second trial function one finds instead

$$\begin{aligned}
d &= 2\,, & A(2) &= 12\pi\ln 2\frac{\ln 2 + \frac{1}{2}}{2\ln 2 - \frac{1}{2}} = 35.178\ldots \\
d &= 3\,, & A(3) &= \frac{8\pi}{3}\frac{(\zeta(2)+2)^{3/2}\,\zeta^{1/2}(2)}{\zeta(2)-1} = 115.934\ldots \\
d &= 4\,, & A(4) &= \frac{\pi^2}{16}\frac{(9\zeta(3) + 24\ln 2)^2}{9\zeta(3) - 12\ln 2} = \pi^2 \times 18.840\ldots\ ,
\end{aligned}$$

where $\zeta(s)$ is Riemann's ζ function.

This can be compared with the exact results $A(2) = 35.10268...$, $A(3) = 113.38350...$ and $A(4) = 16\pi^2$. We see that both variational estimates are better for smaller dimensions, but also that the 1D trial function is much better.

Exercise 28.2

Consider the unphysical field theory in d dimensions, whose action is

$$\mathcal{S}(\phi)=\int \mathrm{d}^d x\left[\frac{1}{2}\phi(x)\,\mathrm{e}^{-\nabla^2}\phi(x)+\frac{1}{4}g\phi^4(x)\right],$$

in which ∇^2 is just the Laplace operator in d dimensions. In this theory the free propagator is in momentum space e^{-p^2}, thus the theory is UV finite in any dimension, and the Feynman diagrams can be calculated algebraically. It is also possible to calculate analytically the contribution of the presumably dominant instanton solution.

Ansatz: Look for a solution of the form $\mathrm{e}^{-\alpha x^2}$. Solution in Bervillier *et al.*, *Phys. Rev.* D17 (1978) 2144.

Exercise 28.3

Generalize the calculations of Chapter 40 to the massless ϕ^{2n} field theory in the dimension in which it is renormalizable:

$$\mathcal{S}(\phi)=\int \mathrm{d}^d x\left[\tfrac{1}{2}\left(\partial_\mu\phi\right)^2+\frac{g}{2n!}\phi^{2n}\right].$$

The eigenvalues of $\mathbf{L}^2$ in d dimensions are $l(l+d-2)$ with a degeneracy $\delta_l(d)$:

$$\delta_l(d)=\frac{(d+2l-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)}.$$

We also recall that if $M(t_1,t_2)$ is of the form:

$$M(t_1,t_2)=\left[-\left(\frac{\mathrm{d}}{\mathrm{d}t_1}\right)^2+1-\frac{\lambda\left(\lambda+1\right)\omega^2}{\cosh^2\omega t_1}\right]\delta\left(t_1-t_2\right),$$

in the notation of equation (39.36) we have:

$$\det(M+\varepsilon)(M_0+\varepsilon)^{-1}=\frac{\Gamma\left(1+z\right)\Gamma\left(z\right)}{\Gamma\left(1+\lambda+z\right)\Gamma(z-\lambda)},\tag{44.90}$$

with

$$z=\sqrt{1+\varepsilon}/\omega\,.\tag{44.91}$$

Solution. We first note that the dimension in which the theory is renormalizable, and thus the classical theory scale invariant, is $d=2n/(n-1)$. The calculations then are a simple generalization of the ϕ^4 field theory.

The dominant instanton still corresponds to a field depending only on a radial coordinate r. We set

$$\phi(x)=(-g)^{-1/(2n-2)}f(r).$$

The classical action becomes

$$\begin{aligned}\mathcal{S}(\phi)&=\frac{S_d}{(-g)^{1/(n-1)}}\Sigma(f)\\ \Sigma(f)&=\int_0^\infty r^{d-1}\mathrm{d}r\left(\tfrac{1}{2}f'^2(r)-\tfrac{1}{2(n-1)^2}f^{2n}(r)\right),\end{aligned}$$

with $S_d = 2\pi^{d/2}/\Gamma(d/2)$. The action is invariant under the scale transformation $f(r) \mapsto \lambda^{(d-2)/2} f(\lambda r)$. Therefore we set $r = \mathrm{e}^t$ and

$$f(r) = b^{1/2}(d)\,\mathrm{e}^{-(d-2)t/2}\,h(t), \qquad b(d) = \left[(d-2)^2(2n!)/8\right]^{(d-2)/2}. \tag{44.92}$$

The action then becomes

$$\Sigma(h) = \tfrac{1}{2} b(d) \int_{-\infty}^{+\infty} \mathrm{d}t \left[\dot{h}^2(t) + \tfrac{1}{(n-1)^2}\left(h^2(t) - h^{2n}(t)\right)\right].$$

We recognize a classical action which we have already encountered in the exercises of Chapter 39 devoted to simple quantum mechanics. The instanton solution of the equation of motion is

$$|h|^{n-1}(t) = \frac{1}{\cosh[(t-t_0)]}.$$

The classical action $\mathcal{S}_c$ is

$$\begin{aligned} \mathcal{S}_c &= A(d)/(-g)^{1/(n-1)}, \\ A(d) &= 2^{d-3} S_d\, b(d)(d-2)\frac{\Gamma^2(d/2)}{\Gamma(d)} = 2(2n!)^{(d-2)/2}\left(\frac{\pi}{2}\right)^{d/2}(d-2)^{d-1}\frac{\Gamma(d/2)}{\Gamma(d)}. \end{aligned}$$

If now more generally we change variables in the action

$$\mathbf{x} \mapsto\ t = \ln r\,, \qquad \mathbf{n} = \mathbf{x}/r\,,$$

and allow the field (44.92) to depend on $\mathbf{n}$ we find a scaled action

$$\Sigma(h) = \tfrac{1}{2} b(d) \int \mathrm{d}\Omega \mathrm{d}t \left[\dot{h}^2(t) + h\mathbf{L}^2 h + \tfrac{1}{(n-1)^2} h^2(t) - \tfrac{1}{(n-1)^2} h^{2n}(t)\right],$$

where L^2 is the angular momentum operator with eigenvalues $l(l+2/(n-1))$. The second derivative $\mathbf{M}$ of the action at the saddle point at angular momentum l is proportional to:

$$\mathbf{M}_l = -\mathrm{d}_t^2 + \left(l + 1/(n-1)\right)^2 - \frac{n(2n-1)}{\cosh^2(t)}.$$

In the notation (44.90) we find

$$\lambda = n/(n-1) \qquad z = \left[\left(l + 1/(n-1)\right)^2 + \varepsilon\right]^{1/2}.$$

We observe that $\mathbf{M}$ has vanishing eigenvalues for $l = 0$ and $l = 1$, as in the simple ϕ^4 example, corresponding to the dilatation and translation modes respectively. One has to introduce collective coordinates, and calculate the jacobian. One important difference between the $n = 2$ and $n > 2$ case is the absence of one-loop renormalization constant of the coupling constant. The remaining part of the calculation can be found in

E. Brézin, J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* D15 (1977) 1544, where results are more generally given for the $\left(\phi^2\right)^n$ $O(N)$ symmetric theory.

44.29 Multi-instantons in Quantum Mechanics : Chapter 43

Exercise 29.1

Complete the calculation of Appendix 41.3 by evaluating explicitly the ground state energy at leading order for the hamiltonian H:

$$H = \tfrac{1}{2}\left[-\Omega(\mathrm{d}/\mathrm{d}q)^2 + \tfrac{1}{4}\Omega^{-1}\left(E'(q)\right)^2 - \tfrac{1}{2}E''(q)\right],$$

in which Ω is the expansion parameter and $E(q) = q^2 - 2q^3/3$.

Solution. The corresponding euclidean action is

$$\mathcal{S}(q) = \frac{1}{2\Omega}\int \mathrm{d}t\left[\dot{q}^2(t) + \tfrac{1}{4}\left(E'(q)\right)^2 - \tfrac{1}{2}\Omega E''(q)\right].$$

At leading order for Ω small $E''(q)$ can be neglected in the classical equation of motion. However, evaluated for the classical solution, it will yield a multiplicative factor in the result. The potential $\frac{1}{4}[E'(q)]^2$ is exactly the double-well potential. Thus the classical solution is the same. The subtle point is that, due to the $E''(q)$ contribution the wells are not really degenerate and the leading term in the ground state energy is a instanton-antiinstanton pair. A similar situation has been examined in Section 43.3 and the calculation is quite similar. Here we have $\omega = 1$ but we have to add to the instanton action terms corresponding to the values of E'' near the well multiplied by the time spent in the well. In the notation of Section 43.3 ($\Omega = g$) we find

$$\begin{aligned}\left\{\operatorname{tr}\mathrm{e}^{-\beta H}\right\}_{(n)} = \frac{\beta}{n}\,\mathrm{e}^{-\beta/2}\,\frac{\mathrm{e}^{-n/3g}}{(\pi g)^n}\int_{\theta_i,\varphi_i\ge 0}\delta\left(\sum_i \theta_i + \varphi_i - \beta\right)\\ \times\exp\left\{\sum_{i=1}^{n}\left[\frac{1}{2}(\theta_i - \varphi_i) + \frac{2}{g}\left(\mathrm{e}^{-\theta_i} + \mathrm{e}^{-\varphi_i}\right)\right]\right\}.\end{aligned} \tag{44.93}$$

The effect of these additional terms is simply to shift by $\pm\frac{1}{2}$ the argument of the function $I(s)$. The spectral equation then reads

$$\left(-\frac{2}{g}\right)^{2E}\Gamma(-E)\Gamma(1-E)\frac{\mathrm{e}^{-1/3g}}{2\pi} = -1\,. \tag{44.94}$$

Let us extract from this equation the ground state energy

$$E \sim \frac{\mathrm{e}^{-1/3g}}{2\pi}\bigl(1 + O(g)\bigr).$$

By expanding to next order in E we also obtain the instanton contribution to the imaginary part of E at leading order

$$\operatorname{Im} E(g) \sim \frac{\mathrm{e}^{-2/3g}}{2\pi}\,,$$

a result consistent with the property that the leading contribution to E is already of order $\mathrm{e}^{-1/3g}$. This result is peculiar to the ground state. Excited states have a non-trivial perturbative expansion. Note that at this order the equation seems to indicate that starting from the first excited states the energies are pairwise degenerate in perturbation theory and that the instanton contribution corresponds to the splitting between states in the two wells. Actually this is a feature of the leading order. There is no symmetry between the wells and the degeneracy is lifted in perturbation theory. In the Γ-functions E is replaced by two different functions. The calculation of the instanton contribution, which is proportional to $\mathrm{e}^{-1/3g}/g^{2N+1}$, requires the knowledge of these functions at order g.